Chapter

2

Complex Numbers

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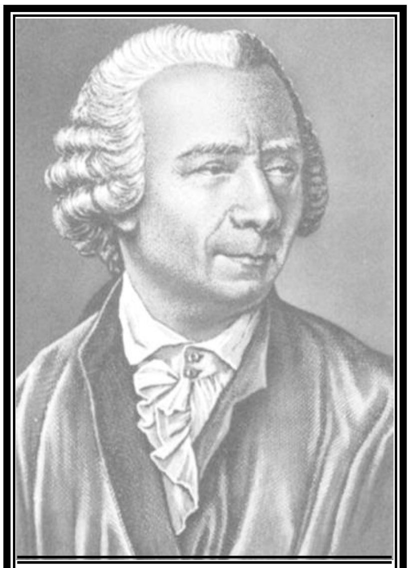


Leonhard Euler

The fact that square root of a negative number does not exist in the real number system was recognized by the Greeks. But the credit goes to the Indian mathematician Mahavira (850 A.D.) *Who first stated this difficulty clearly "He mentions in his work* "Ganitasara Sangraha" *as in the nature of things a negative (quantity) is not a square (quantity), it has, therefore no square root".* Bhaskara, *another Indian mathematician, also writes in this work* 'Bijaganita' *written in* 1150 A.D.

Euler *was the first to introduce the symbol* i *for* $\sqrt{-1}$ *and* W.R. Hamilton *(about 1830 A.D.) regarded the complex number a + ib as an ordered pair of real numbers (a, b), thus giving it a purely mathematical definition and avoiding use of the so called "Imaginary numbers".*

Complex Numbers

2.1 Introduction

Number system consists of real numbers $(-5, 7, \frac{1}{3}, \sqrt{3}............etc.)$ and imaginary numbers $(\sqrt{-5}, \sqrt{-9}$ ....etc.) If we combine these two numbers by some mathematical operations, the resulting number is known as Complex Number *i.e.*, "Complex Number is the combination of real and imaginary numbers".

(1) **Basic concepts of complex number**

(i) **General definition :** A number of the form $x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$ is called a complex number so the quantity $\sqrt{-1}$ is denoted by 'i' called iota thus $i = \sqrt{-1}$.

A complex number is usually denoted by z and the set of complex number is denoted by c

i.e., $c = \{x + iy : x \in R, y \in R, i = \sqrt{-1}\}$

For example, $5 + 3i, -1 + i, 0 + 4i, 4 + 0i$ etc. are complex numbers.

Note :
- Euler was the first mathematician to introduce the symbol i (iota) for the square root of – 1 with property $i^2 = -1$. He also called this symbol as the imaginary unit.
- Iota (i) is neither 0, nor greater than 0, nor less than 0.
- The square root of a negative real number is called an imaginary unit.
- For any positive real number a, we have $\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}$
- $i\sqrt{-a} = -\sqrt{a}$.
- The property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is valid only if at least one of a and b is non-negative. If a and b are both negative then $\sqrt{a}\sqrt{b} = -\sqrt{ab}$.
- If $a < 0$ then $\sqrt{a} = \sqrt{|a|}\, i$.

(2) **Integral powers of iota (i) :** Since $i = \sqrt{-1}$ hence we have $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$. To find the value of i^n $(n > 4)$, first divide n by 4. Let q be the quotient and r be the remainder.

i.e., $n = 4q + r$ where $0 \le r \le 3$

$i^n = i^{4q+r} = (i^4)^q . (i)^r = (1)^q . (i)^r = i^r$

In general we have the following results $i^{4n} = 1, i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i$, where n is any integer.

In other words, $i^n = (-1)^{n/2}$ if n is even integer and $i^n = (-1)^{n-1/2} i$ if n is odd integer.

The value of the negative integral powers of i are found as given below :

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i, i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,\ i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = \frac{i}{1} = i,\ i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

Important Tips

- *The sum of four consecutive powers of i is always zero i.e.,* $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0,\ n \in I.$
- $i^n = 1, i, -1, -i,$ *where n is any integer.*
- $(1+i)^2 = 2i,\ (1-i)^2 = -2i$
- $\frac{1+i}{1-i} = i,\ \frac{1-i}{1+i} = -i,\ \frac{2i}{i-1} = 1-i$

Example: 1 If $i^2 = -1$, then the value of $\sum_{n=1}^{200} i^n$ is **[MP PET 1996]**

(a) 50 (b) – 50 (c) 0 (d) 100

Solution: (c) $\sum_{n=1}^{200} i^n = i + i^2 + i^3 + ... + i^{200} = \frac{i(1-i^{200})}{1-i}$ (since G.P.) $= \frac{i(1-1)}{1-i} = 0.$

Example: 2 If $i = \sqrt{-1}$ and n is a positive integer, than $i^n + i^{n+1} + i^{n+2} + i^{n+3} =$ **[Rajasthan PET 2001; Karnataka CET 1994]**

(a) 1 (b) i (c) i^n (d) 0

Solution: (d) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n(1 + i + i^2 + i^3) = i^n(1 + i - 1 - i) = o.$

Trick: Since the sum of four consecutive powers of i is always zero.

$\Rightarrow i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0,\ \ n \in I.$

Example: 3 If $\left(\frac{1+i}{1-i}\right)^x = 1$ then **[AIEEE 2003; Rajasthan PET 2003]**

(a) $x = 4n$, where n is any positive integer (b) $x = 2n$, where n is any positive integer

(c) $x = 4n + 1$, where n is any positive integer (d) $x = 2n + 1$, where n is any positive integer

Solution: (a) $\frac{1+i}{1-i} = \frac{1+i}{1-i}.\frac{1+i}{1+i} \Rightarrow \left(\frac{1+i}{1-i}\right)^x = 1 \Rightarrow i^x = 1 \Rightarrow x = 4n,\ n \in I^+.$

Example: 4 $1 + i^2 + i^4 + i^6 + ... + i^{2n}$ is **[EAMCET 1980]**

(a) Positive (b) Negative (c) Zero (d) Can not be determined

Solution: (d) $1 + i^2 + i^4 + i^6 + ... + i^{2n} = \sum_{k=0}^{n} i^{2k} = 1$ or – 1 (which is depend upon the value of n).

Example: 5 If $x = 3 + i$, then $x^3 - 3x^2 - 8x + 15 =$ **[UPSEAT 2003]**

(a) 6 (b) 10 (c) – 18 (d) – 15

Solution: (d) Given that; $x - 3 = i \Rightarrow (x-3)^2 = i^2 \Rightarrow x^2 - 6x + 10 = 0$

Now, $x^3 - 3x^2 - 8x + 15 = x(x^2 - 6x + 10) + 3(x^2 - 6x + 10) - 15 = 0 + 0 - 15 = -15.$

Example: 6 The complex number $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}, (n \in Z)$ is equal to

(a) 0 (b) 2 (c) $[1 + (-1)^n].i^n$ (d) None of these

Solution: (d) $(1+i)^{2n} = ((1+i)^2)^n = (1 + i^2 + 2i)^n = (1 - 1 + 2i)^n = 2^n i^n$

$(1-i)^{2n} = ((1-i)^2)^n = (1 + i^2 - 2i)^n = (1 - 1 - 2i)^n = (-2)^n i^n$

$$\therefore \frac{2^n}{(1-i)2^n}+\frac{(1+i)^{2n}}{2^n}=\frac{2^n}{(-2)^n i^n}+\frac{2^n i^n}{2^n}=\frac{1}{(-1)^n i^n}+i^n=\frac{1+(-1)^n i^{2n}}{(-1)^n i^n}=\frac{1+(-1)^n (i^2)^n}{(-1)^n i^n}$$

$$=\frac{1+(-1)^n(-1)^n}{(-1)^n i^n}=\frac{1+(-1)^{2n}}{(-1)^n i^n}=\frac{1+1}{(-1)^n i^n}=\frac{2}{(-1)^n i^n}.$$

2.2 Real and Imaginary Parts of a Complex Number

If x and y are two real numbers, then a number of the form $z = x + iy$ is called a complex number. Here 'x' is called the real part of z and 'y' is known as the imaginary part of z. The real part of z is denoted by $Re(z)$ and the imaginary part by $\text{Im}(z)$.

If $z = 3 - 4i$, then $\text{Re}(z) = 3$ and $\text{Im}(z) = -4$.

Note :
- ❑ A complex number z is purely real if its imaginary part is zero *i.e.*, $\text{Im}(z) = 0$ and purely imaginary if its real part is zero *i.e.*, $\text{Re}(z) = 0$.
- ❑ i can be denoted by the ordered pair (0,1).
- ❑ The complex number (a, b) can also be split as $(a, 0) + (0, 1)\ (b, 0)$.

Important Tips

- ☞ *A complex number is an imaginary number if and only if its imaginary part is non-zero. Here real part may or may not be zero.*
- ☞ *All purely imaginary numbers except zero are imaginary numbers but an imaginary number may or not be purely imaginary.*
- ☞ *A real number can be written as a + i.0, therefore every real number can be considered as a complex number whose imaginary part is zero. Thus the set of real number (R) is a proper subset of the complex number (C) i.e., R ⊂ C.*
- ☞ *Complex number as an ordered pair : A complex number may also be defined as an ordered pair of real numbers and may be denoted by the symbol (a,b). For a complex number to be uniquely specified, we need two real numbers in particular order.*

2.3 Algebraic Operations with Complex Numbers

Let two complex numbers $z_1 = a + ib$ and $z_2 = c + id$

Addition : $(a+ib)+(c+id)=(a+c)+i(b+d)$

Subtraction : $(a+ib)-(c+id)=(a-c)+i(b-d)$

Multiplication : $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$

Division : $\dfrac{a+ib}{c+id}$ (when at least one of c and d is non-zero)

$$\frac{a+ib}{c+id}=\frac{(a+ib)}{(c+id)}\cdot\frac{(c-id)}{(c-id)} \quad \text{(Rationalization)}$$

$$\frac{a+ib}{c+id}=\frac{(ac+bd)}{c^2+d^2}+\frac{i(bc-ad)}{c^2+d^2}.$$

Properties of algebraic operations with complex numbers : Let z_1, z_2 and z_3 are any complex numbers then their algebraic operation satisfy following operations:

(i) Addition of complex numbers satisfies the commutative and associative properties

i.e., $z_1 + z_2 = z_2 + z_1$ and $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(ii) Multiplication of complex number satisfies the commutative and associative properties.

i.e., $z_1 z_2 = z_2 z_1$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

(iii) Multiplication of complex numbers is distributive over addition

i.e., $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ and $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$.

Note :
- ❑ $0 = 0 + 0i$ is the identity element for addition.
- ❑ $1 = 1 + 0i$ is the identity element for multiplication.
- ❑ The additive inverse of a complex number $z = a + ib$ is $-z$ (*i.e.* $-a-ib$).
- ❑ For every non-zero complex number z, the multiplicative inverse of z is $\frac{1}{z}$.

Example: 7 $\frac{1-2i}{2+i} + \frac{4-i}{3+2i} =$ **[Rajasthan PET 1987]**

(a) $\frac{24}{13} + \frac{10}{13} i$ (b) $\frac{24}{13} - \frac{10}{13} i$ (c) $\frac{10}{13} + \frac{24}{13} i$ (d) $\frac{10}{13} - \frac{24}{13} i$

Solution: (d) $\frac{1-2i}{2+i} + \frac{4-i}{3+2i} = = \frac{(1-2i)(3+2i)+(4-i)(2+i)}{(2+i)(3+2i)} = \frac{50-120i}{65} = \frac{10}{13} - \frac{24}{13} i.$

Example: 8 $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ **[Roorkee 1979; Rajasthan PET 1999]**

(a) $\frac{1}{2} + \frac{9}{2} i$ (b) $\frac{1}{2} - \frac{9}{2} i$ (c) $\frac{1}{4} - \frac{9}{4} i$ (d) $\frac{1}{4} + \frac{9}{4} i$

Solution: (d) $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right) = \left[\frac{1+2i}{1^2+2^2} + \frac{3-3i}{1^2+1^2}\right]\left[\frac{6-16+12i+8i}{2^2+4^2}\right] = \left(\frac{2+4i+15-15i}{10}\right)\left(\frac{-1+2i}{2}\right)$

$= \frac{(17-11i)(-1+2i)}{20} = \frac{5+45i}{20} = \frac{1}{4} + \frac{9}{4} i.$

Example: 9 The real value of θ for which the expression $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number, is

(a) $n\pi + \pi/2$ (b) $n\pi - \pi/2$ (c) $n\pi \pm \pi/2$ (d) None of these

Solution: (c) Given that $\frac{1+i\cos\theta}{1-2i\cos\theta} = \frac{(1+i\cos\theta)(1+2i\cos\theta)}{(1-2i\cos\theta)(1+2i\cos\theta)} = \left[\frac{(1-2\cos^2\theta)}{(1+4\cos^2\theta)}\right] + i\left[\frac{3\cos\theta}{1+4\cos^2\theta}\right]$

Since $\text{Im}(z) = 0$, then $3\cos\theta = 0 \Rightarrow \theta = n\pi \pm \pi/2$.

2.4 Equality of Two Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if their real parts and imaginary parts are separately equal.

i.e., $z_1 = z_2 \Rightarrow x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$.

Thus , one complex equation is equivalent to two real equations.

Note : ❑ A complex number $z = x + iy = 0$ *iff* $x = 0, y = 0$.

❑ The complex number do not possess the property of order *i.e.*, $(a + ib) < (or) > (c + id)$ is not defined. For example, the statement $9 + 6i > 3 + 2i$ makes no sense.

Example: 10 Which of the following is correct

(a) $6 + i > 8 - i$ (b) $6 + i > 4 - i$ (c) $6 + i > 4 + 2i$ (d) None of these

Solution: (d) Because, inequality is not applicable for a complex number.

Example: 11 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then **[MP PET 2000; IIT 1998]**

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$ (c) $x = 0, y = 3$ (d) $x = 0, y = 0$

Solution: (d) $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix}$ Applying $C_2 \to C_2 + 3iC_3$

$\begin{vmatrix} 6i & 0 & 1 \\ 4 & 0 & -1 \\ 20 & 0 & i \end{vmatrix} = 0 = 0 + 0\,i$, Equating real and imaginary parts $x = 0$, $y = 0$

Example: 12 The real values of x and y for which the equation $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi)$ is satisfied, are **[Roorkee**

(a) $x = 2, y = 3$ (b) $x = -2, y = \frac{1}{3}$ (c) Both (a) and (b) (d) None of these

Solution: (c) Given equation $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi) \Rightarrow (x^4 - 3x^2) + i(2x - 3y) = 4 - 5i$

Equating real and imaginary parts, we get

$x^4 - 3x^2 = 4$ .....(i) and $2x - 3y = -5$ .....(ii)

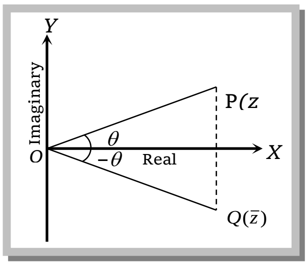
Form (i) and (ii), we get $x = \pm 2$ and $y = 3, \frac{1}{3}$.

Trick: Put $x = 2, y = 3$ and then $x = -2, y = \frac{1}{3}$, we see that they both satisfy the given equation.

2.5 Conjugate of a Complex Number

(1) **Conjugate complex number :** If there exists a complex number $z = a + ib$, $(a, b) \in R$, then its conjugate is defined as $\bar{z} = a - ib$.

Hence, we have $\text{Re}(z) = \frac{z + \bar{z}}{2}$ and $\text{Im}(z) = \frac{z - \bar{z}}{2i}$. Geometrically, the conjugate of z is the reflection or point image of z in the real axis.



(2) **Properties of conjugate :** If z, z_1 and z_2 are existing complex numbers, then we have the following results:

(i) $\overline{(\bar{z})} = z$ (ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (iv) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$, In general $\overline{z_1.z_2.z_3.....z_n} = \bar{z}_1.\bar{z}_2.\bar{z}_3.....\bar{z}_n$

(v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$ (vi) $(\overline{z})^n = (\overline{z^n})$

(vii) $z + \overline{z} = 2\,\text{Re}(z) = 2\,\text{Re}(\overline{z}) =$ purely real (viii) $z - \overline{z} = 2i\,\text{Im}(z) =$ purely imaginary

(ix) $z\overline{z} =$ purely real (x) $z_1\overline{z}_2 + \overline{z}_1 z_2 = 2\,\text{Re}(z_1\overline{z}_2) = 2\,\text{Re}(\overline{z}_1 z_2)$

(xi) $z - \overline{z} = 0$ *i.e.*, $z = \overline{z} \Leftrightarrow z$ is purely real *i.e.*, $\text{Im}(z) = 0$

(xii) $z + \overline{z} = 0$ *i.e.*, $z = -\overline{z} \Leftrightarrow$ either $z = 0$ or z is purely imaginary *i.e.*, $\text{Re}(z) = 0$

(xiii) $z_1 = z_2 \Leftrightarrow \overline{z}_1 = \overline{z}_2$ (xiv) $z = 0 \Leftrightarrow \overline{z} = 0$

(xv) $z\overline{z} = 0 \Leftrightarrow z = 0$ (xvi) If $w = f(z)$ then $\overline{w} = f(\overline{z})$ (xvii) $\overline{re^{i\theta}} = re^{-i\theta}$

Important Tips

- *Complex conjugate is obtained by just changing the sign of i.*
- *Conjugate of* $i = -i$
- *Conjugate of* $iz = -i\overline{z}$
- $(z_1 + z_2)$ *and* $(z_1 . z_2)$ *real* $\Leftrightarrow z_1 = \overline{z}_2$ *or* $z_2 = \overline{z}_1$
- $\overline{z_1\overline{z}_2} = \overline{z}_1 z_2$

(3) **Reciprocal of a complex number :** For an existing non-zero complex number $z = a + ib$, the reciprocal is given by $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}$ *i.e.*, $z^{-1} = \frac{1}{a+ib} \Rightarrow \frac{a-ib}{a^2+b^2} = \frac{\text{Re}(z)}{|z|^2} + \frac{i[-\text{Im}(z)]}{|z|^2} = \frac{\overline{z}}{|z|^2}$.

Example: 13 If the conjugate of $(x+iy)(1-2i)$ be $1+i$, then **[MP PET 1996]**

(a) $x = \frac{1}{5}$ (b) $y = \frac{3}{5}$ (c) $x + iy = \frac{1-i}{1-2i}$ (d) $x - iy = \frac{1-i}{1+2i}$

Solution: (c) Given that $\overline{(x+iy)(1-2i)} = 1 + i \Rightarrow x - iy = \frac{1+i}{1+2i} \Rightarrow x + iy = \frac{1-i}{1-2i}$

Example: 14 For the complex number z, one from $z + \overline{z}$ and $z\overline{z}$ is

(a) A real number (b) An imaginary number
(c) Both are real numbers (d) Both are imaginary numbers

Solution: (c) Here $z + \overline{z} = (x+iy) + (x-iy) = 2x$ (Real) and $z\overline{z} = (x+iy)(x-iy) = x^2 + y^2$ (Real).

Example: 15 The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for **[IIT 1988]**

(a) $x = n\pi$ (b) $x = \left(n + \frac{1}{2}\right)\pi$ (c) $x = 0$ (d) No value of x

Solution: (d) $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other if $\sin x = \cos x$ and $\cos 2x = \sin 2x$

or $\tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, ...$ (i) and $\tan 2x = 1 \Rightarrow 2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, ...$ or $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, ...$ (ii) There exists no value of x common in (i) and (ii). Therefore there is no value of x for which the given complex numbers are conjugate.

Example: 16 The conjugate of complex number $\frac{2-3i}{4-i}$ is **[MP PET 2004]**

(a) $\frac{3i}{4}$ (b) $\frac{11+10i}{17}$ (c) $\frac{11-10i}{17}$ (d) $\frac{2+3i}{4i}$

Solution: (b) $\frac{2-3i}{4-i}=\frac{(2-3i)(4+i)}{(4-i)(4+i)}=\frac{8+3-12i+2i}{16+1}=\frac{11-10i}{17}\Rightarrow$ Conjugate $=\frac{11+10i}{17}$.

Example: 17 The real part of $(1-\cos\theta+2i\sin\theta)^{-1}$ is **[Karnataka CET 2001; IIT 1978, 86]**

(a) $\frac{1}{3+5\cos\theta}$ (b) $\frac{1}{5-3\cos\theta}$ (c) $\frac{1}{3-5\cos\theta}$ (d) $\frac{1}{5+3\cos\theta}$

Solution: (c) $\{(1-\cos\theta)+i.2\sin\theta\}^{-1}=\left\{2\sin^2\frac{\theta}{2}+i.4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}^{-1}=\left(2\sin\frac{\theta}{2}\right)^{-1}\left\{\sin\frac{\theta}{2}+i.2\cos\frac{\theta}{2}\right\}^{-1}$

$$=\left(2\sin\frac{\theta}{2}\right)^{-1}.\frac{1}{\sin\frac{\theta}{2}+i.2\cos\frac{\theta}{2}}\times\frac{\sin\frac{\theta}{2}-i.2\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}-i.2\cos\frac{\theta}{2}}=\frac{\sin\frac{\theta}{2}-i.2\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2}+4\cos^2\frac{\theta}{2}\right)}$$

Hence, real part $=\frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(1+3\cos^2\frac{\theta}{2}\right)}=\frac{1}{2\left(1+3\cos^2\frac{\theta}{2}\right)}=\frac{1}{5+3\cos\theta}$.

Example: 18 The reciprocal of $3+\sqrt{7}i$ is

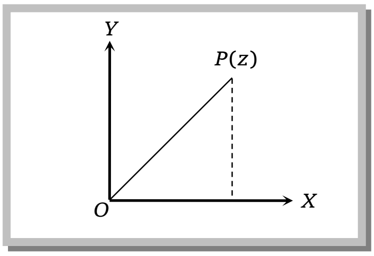
(a) $\frac{3}{4}-\frac{\sqrt{7}}{4}i$ (b) $3-\sqrt{7}i$ (c) $\frac{3}{16}-\frac{\sqrt{7}}{16}i$ (d) $\sqrt{7}+3i$

Solution: (c) $\frac{1}{3+\sqrt{7}i}=\frac{1}{3+\sqrt{7}i}.\frac{3-\sqrt{7}i}{3-\sqrt{7}i}=\frac{3-\sqrt{7}i}{9+7}=\frac{3-\sqrt{7}i}{16}=\frac{3}{16}-\frac{\sqrt{7}}{16}i$.

2.6 Modulus of a Complex Number

Modulus of a complex number $z=a+ib$ is defined by a positive real number given by $|z|=\sqrt{a^2+b^2}$, where a, b real numbers. Geometrically $|z|$ represents the distance of point P (represented by z) from the origin,

i.e. $|z| = OP$.



If $|z| = 0$, then z is known as zero modular complex number and is used to represent the origin of reference plane.

If $|z| = 1$ the corresponding complex number is known as **unimodular complex number.** Clearly z lies on a circle of unit radius having centre (0, 0).

Note : ❑ In the set C of all complex numbers, the order relation is not defined. As such $z_1>z_2>$ or $z_1<z_2$ has no meaning. But $|z_1|>|z_2|$ or $|z_1|<|z_2|$ has got its meaning since $|z_1|$ and $|z_2|$ are real numbers.

Properties of modulus

(i) $|z|\geq 0\Rightarrow|z|=0$ iff $z=0$ and $|z|>0$ iff $z\neq 0$.

(ii) $-|z|\leq \text{Re}(z)\leq|z|$ and $-|z|\leq \text{Im}(z)\leq|z|$

(iii) $|z|=|\bar{z}|=|-z|=|-\bar{z}|=|zi|$

(iv) $z\bar{z} = |z|^2 = |\bar{z}|^2$

(v) $|z_1 z_2| = |z_1||z_2|$. In general $|z_1 z_2 z_3 ......z_n| = |z_1||z_2||z_3|....|z_n|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $(z_2 \neq 0)$

(vii) $|z^n| = |z|^n, n \in N$

(viii) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$ or $|z_1|^2 + |z_2|^2 \pm 2\,\text{Re}(z_1\bar{z}_2)$

(ix) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary or $\text{Re}\left(\frac{z_1}{z_2}\right) = 0$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$ (Law of parallelogram)

(xi) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$, where $a, b \in R$.

Important Tips

- *Modulus of every complex number is a non-negative real number.*
- $|z| = 0$ *iff* $z = 0$ *i.e.,* $\text{Re}(z) = \text{Im}(z) = 0$
- $|z| \geq |\text{Re}(z)| \geq \text{Re}(z)$ *and* $|z| \geq |\text{Im}(z)| \geq \text{Im}(z)$
- $|z| = 1 \Leftrightarrow \bar{z} = \frac{1}{z}$
- $\left|\frac{z}{\bar{z}}\right| = 1$
- $\frac{z}{|z|}$ *is always a unimodular complex number if* $z \neq 0$
- $\frac{z}{\bar{z}}$ *is always a unimodular complex number if* $z \neq 0$
- $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}|z|$
- $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
  Thus $|z_1| + |z_2|$ *is the greatest possible value of* $|z_1 + z_2|$ *and* $||z_1| - |z_2||$ *is the least possible value of* $|z_1 + z_2|$
- *If* $\left|z + \frac{1}{z}\right| = a$, *the greatest and least values of* $|z|$ *are respectively* $\frac{a + \sqrt{a^2 + 4}}{2}$ *and* $\frac{-a + \sqrt{a^2 + 4}}{2}$
- $|z_1 + \sqrt{z_1^2 - z_2^2}| + |z_1 - \sqrt{z_1^2 - z_2^2}| = |z_1 + z_2| + |z_1 - z_2|$

Example: 19 $\left|(1+i)\frac{(2+i)}{(3+i)}\right| =$ **[MP PET 1995, 99; Rajasthan PET 1998]**

(a) – 1/2 (b) 1/2 (c) 1 (d) – 1

Solution (c) $z = \frac{(1+i)(2+i)}{(3+i)} = \frac{1+3i}{3+i} \times \frac{3-i}{3-i} = \frac{3+4i}{5} \Rightarrow |z| = 1$

Trick : $|z| = \frac{|z_1||z_2|}{|z_3|} = \frac{\sqrt{2}.\sqrt{5}}{\sqrt{10}} = 1$

Example: 20 If α and β are different complex numbers with $|\beta| = 1$, then $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$ is equal to

(a) 0 (b) 1/2 (c) 1 (d) 2

Solution (c) $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{\beta - \alpha}{\beta\bar{\beta} - \bar{\alpha}\beta}\right| = \left|\frac{\beta - \alpha}{\beta(\bar{\beta} - \bar{\alpha})}\right|, = \frac{1}{|\beta|}\left|\frac{\beta - \alpha}{(\overline{\beta - \alpha})}\right| = \frac{1}{|\beta|} = 1$ $\{\because |z| = |\bar{z}|\}$

Example: 21 For any complex number z, maximum value of $|z|-|z-1|$ is

(a) 0 (b) 1 (c) 3/2 (d) None of these

Solution (b) We know that $|z_1-z_2|\geq |z_1|-|z_2|$

$\therefore |z|-|z-1|\leq |z-(z-1)|$ or $|z|-|z-1|\leq 1$, $\therefore$ Maximum value of $|z|-|z-1|$ is 1.

Example 22 If $z=x+iy$ and $iz^2-\bar{z}=0$, then $|z|$ is equal to **[Bihar CEE 1998]**

(a) 1 (b) 0 or 1 (c) 1 or 2 (d) 2

Solution: (b) $iz^2=\bar{z}\Rightarrow |iz^2|=|\bar{z}|\Rightarrow |z|^2=|z|\Rightarrow |z|(|z|-1)=0\Rightarrow |z|=0$ or $|z|=1$

Example: 23 For $x_1,x_2,y_1,y_2\in R$, if $0<x_1<x_2, y_1=y_2$ and $z_1=x_1+iy_1, z_2=x_2+iy_2$, and $z_3=\frac{1}{2}(z_1+z_2)$, then z_1,z_2 and z_3 satisfy **[Roorkee 1991]**

(a) $|z_1|=|z_2|=|z_3|$ (b) $|z_1|<|z_2|<|z_3|$ (c) $|z_1|>|z_2|>|z_3|$ (d) $|z_1|<|z_3|<|z_2|$

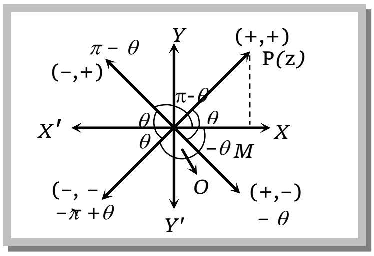
Solution: (d) $0<x_1<x_2, y_1=y_2$ (Given)

$$|z_1|=\sqrt{x_1^2+y_1^2}, |z_2|=\sqrt{x_2^2+y_2^2}\Rightarrow |z_2|>|z_1|\Rightarrow |z_3|=\frac{|z_1+z_2|}{2}=\sqrt{\left(\frac{x_1+x_2}{2}\right)^2+\left(\frac{y_1+y_2}{2}\right)^2}$$

$\sqrt{\left(\frac{x_1+x_2}{2}\right)^2+y_1^2}<|z_2|>|z_1|$. Hence, $|z_1|<|z_3|<|z_2|$

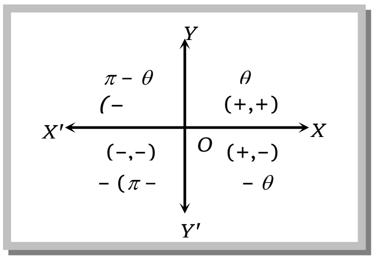
2.7 Argument of a Complex Number

Let $z=a+ib$ be any complex number. If this complex number is represented geometrically by a point P, then the angle made by the line OP with real axis is known as argument or amplitude of z and is expressed as

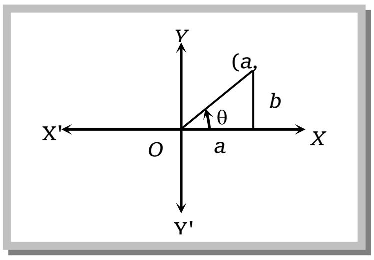


$arg\ (z)=\theta=\tan^{-1}\left(\frac{b}{a}\right)$, $\theta=\angle POM$. Also, argument of a complex number is not unique, since if θ be a value of the argument, so also is $2n\pi+\theta$, where $n\in I$.

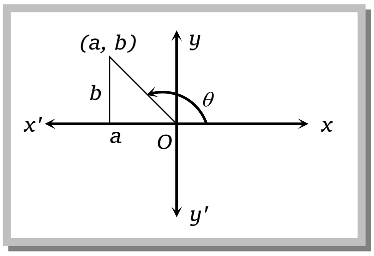
(1) **Principal value of *arg* (z) :** The value θ of the argument, which satisfies the inequality $-\pi<\theta\leq\pi$ is called the principal value of argument. Principal values of argument z will be $\theta, \pi-\theta, -\pi+\theta$ and $-\theta$ according as the point z lies in the 1st, 2nd, 3rd and 4th quadrants respectively, where $\theta=\tan^{-1}\left|\frac{b}{a}\right|=\alpha$ (acute angle). Principal value of argument of any complex number lies between $-\pi<\theta\leq\pi$.



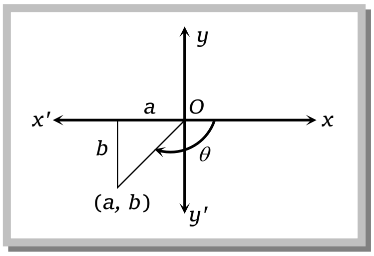
(i) $a,b\in$ First quadrant $a>0, b>0$. $arg\ (z)=\theta=\tan^{-1}\left(\frac{b}{a}\right)$. It is an acute angle and positive.



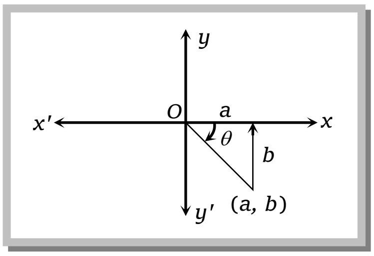
(ii) $(a,b)\in$ Second quadrant, $a<0, b>0$, *arg* $(z)=\theta=\pi-\tan^{-1}\left(\frac{b}{|a|}\right)$. It is an obtuse angle and positive.



(iii) $(a,b)\in$ Third quadrant $a<0, b<0$, *arg* $(z)=\theta=-\pi+\tan^{-1}\left(\frac{b}{a}\right)$. It is an obtuse angle and negative.



(iv) $(a,b) \in$ Fourth quadrant $a > 0, b < 0$, *arg* $(z) = \theta = -\tan^{-1}\left(\frac{|b|}{a}\right)$. It is an acute angle and negative.



Quadrant	x	y	arg(z)	Interval of θ
I	+	+	θ	$0 < \theta < \pi/2$
II	–	+	$\pi - \theta$	$\pi/2 < \theta < \pi$
III	–	–	$-(\pi - \theta)$	$-\pi < \theta < -\pi/2$
IV	+	–	$-\theta$	$-\pi/2 < \theta < 0$

Note :

- Argument of the complex number 0 is not defined.
- Principal value of argument of a purely real number is 0 if the real number is positive and is π if the real number is negative.
- Principal value of argument of a purely imaginary number is $\pi/2$ if the imaginary part is positive and is $-\pi/2$ if the imaginary part is negative.

(2) **Properties of arguments**

(i) $arg\ (z_1 z_2) = arg(z_1) + arg(z_2) + 2k\pi$, $(k = 0$ or 1 or – 1)

In general $arg\,(z_1 z_2 z_3 \ldots\ldots\ldots z_n) = arg(z_1) + arg(z_2) + arg(z_3) + \ldots\ldots\ldots + arg(z_n) + 2k\pi$, $(k = 0$ or 1 or $-1)$

(ii) $arg\,(z_1 \bar{z}_2) = arg(z_1) - arg(z_2)$

(iii) $arg\left(\frac{z_1}{z_2}\right) = arg\, z_1 - arg\, z_2 + 2k\pi$, $(k = 0$ or 1 or – 1)

(iv) $arg\left(\frac{z}{\bar{z}}\right) = 2 arg\, z + 2k\pi$, $(k = 0$ or 1 or – 1)

(v) $arg(z^n) = n\, arg\, z + 2k\pi$, $(k = 0$ or 1 or – 1)

(vi) If $arg\left(\frac{z_2}{z_1}\right) = \theta$, then $arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$, where $k \in I$

(vii) $arg\,\bar{z} = -arg\, z = arg\frac{1}{z}$

(viii) $arg\,(z - \bar{z}) = \pm\pi/2$

(ix) $arg\,(-z) = arg\,(z) \pm \pi$

(x) $arg\,(z) + arg(\bar{z}) = 0$ or $arg(z) = -arg(\bar{z})$

(xi) $arg(z) - arg(\bar{z}) = \pm\pi$

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = arg(z_1)$ and $\theta_2 = arg(z_2)$

Note : ❑ Proper value of k must be chosen so that R.H.S. of (i), (ii), (iii) and (iv) lies in $(-\pi, \pi)$

❑ The property of argument is same as the property of logarithm.

If *arg* (z) lies between $-\pi$ and π(π inclusive), then this value itself is the principal value of *arg* (z). If not, see whether *arg* (z) $> \pi$ or $\le -\pi$. If $arg(z) > \pi$, go on subtracting 2π until it lies between $-\pi$ and π(π inclusive). The value thus obtained will be the principal value of *arg* (z).

❑ The general value of $arg(\bar{z})$ is $2n\pi - arg(z)$.

Important Tips

☞ *If* $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ *and arg* z_1 *= arg* z_2.

☞ $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow$ *arg* $(z_1) =$ *arg* (z_2) *i.e.,* z_1 *and* z_2 *are parallel.*

☞ $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow$ *arg* $(z_1) -$ *arg* $(z_2) = 2n\pi$, *where n is some integer.*

☞ $|z_1 - z_2| = ||z_1| - |z_2|| \Leftrightarrow arg(z_1) - arg(z_2) = 2n\pi$, *where n is some integer.*

☞ $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow$ *arg* $(z_1) -$ *arg* $(z_2) = \pi/2$.

☞ *If* $|z_1| \le 1, |z_2| \le 1$ *then* (i) $|z_1 + z_2|^2 \le (|z_1| - |z_2|)^2 + (arg\ (z_1) - arg\ (z_2))^2$ (ii) $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 - (arg(z_1) - arg(z_2))^2$

☞ $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$.

☞ $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$.

☞ *If* $|z_1| = |z_2|$ *and* $amp\ (z_1) + amp\ (z_2) = 0$, *then* $z_1 + z_2$ *are conjugate complex numbers of each other.*

☞ $z \ne 0$, $amp\ (z + \bar{z}) = 0$ *or* $\pi; amp(z\bar{z}) = 0; amp\ (z - \bar{z}) = \pm\pi/2$.

☞ *arg* $(1) = 0$, *arg* $(-1) = \pi$; *arg* $(i) = \pi/2$, *arg* $(-i) = -\pi/2$.

☞ *arg* $(z) = \frac{\pi}{4} \Rightarrow \text{Re}(z) = \text{Im}(z)$.

☞ *Amplitude of complex number in I and II quadrant is always positive and in III*[rd] *and IV*[th] *quadrant is always negative.*

☞ *If a complex number multiplied by i (Iota) its amplitude will be increased by* $\pi/2$ *and will be decreased by* $\pi/2$, *if multiplied by –i, i.e.* $arg(iz) = \frac{\pi}{2} + arg(z)$ *and* $arg(-iz) = arg(z) - \frac{\pi}{2}$.

Complex number	Value of argument
+*ve* Re (z)	0
–*ve* Re (z)	π
+*ve* Im (z)	$\pi/2$
–*ve* Im (z)	$3\pi/2$ or $-\pi/2$
– (z)	$\|\theta \pm \pi\|$, *if* θ *is* $-ve$ *and* $+ve$ respectively
(iz)	$\left\{\frac{\pi}{2} + arg(z)\right\}$
–(iz)	$\left\{arg(z) - \frac{\pi}{2}\right\}$

(z^n)	$n.$ *arg* (z)
$(z_1.z_2)$	*arg* (z_1) + *arg* (z_2)
$\left(\frac{z_1}{z_2}\right)$	*arg* (z_1) – *arg* (z_2)

Example: 24 Amplitude of $\left(\frac{1-i}{1+i}\right)$ is **[Rajasthan PET 1996]**

(a) $\frac{-\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Solution: (a) $z = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = -i$

$\Rightarrow \theta = \tan^{-1}\left(\frac{-1}{0}\right) = -\frac{\pi}{2}$ (Since z lies on negative imaginary axis)

Example: 25 If $|z_1| = |z_2|$ and $amp\, z_1 + amp\, z_2 = 0$, then **[MP PET 1999]**

(a) $z_1 = z_2$ (b) $\bar{z}_1 = z_2$ (c) $z_1 + z_2 = 0$ (d) $\bar{z}_1 = \bar{z}_2$

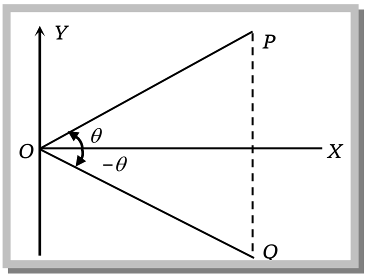
Solution: (b) Let $|z_1| = OP, |z_2| = OQ$

Since *amp* $(z_1) = \theta \Rightarrow$ *amp* $(z_2) = -\theta$

$\because$ Q is point image of P

$\therefore$ $\overline{z_1} = z_2$

Trick : $arg\, z + arg\, \bar{z} = 0$, $\therefore \bar{z}_1$ must be equal to z_2.



Example: 26 Let z, w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and *arg* $zw = \pi$, then *arg* z equals **[AIEEE 2004]**

(a) $5\pi/4$ (b) $\pi/2$ (c) $3\pi/4$ (d) $\pi/4$

Solution: (d) $\bar{z} + i\bar{w} = 0$

$\therefore z = iw \Rightarrow \theta + (\pi/2 + \theta) = \pi$, $\therefore \theta = \pi/4$.

Example: 27 The amplitude of $\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$ **[Karnataka CET 2003]**

(a) $\pi/5$ (b) $2\pi/5$ (c) $\pi/10$ (d) $\pi/15$

Solution: (c) $\sin\frac{\pi}{5} + i(1 - \cos\frac{\pi}{5}) = 2\sin\frac{\pi}{10}.\cos\frac{\pi}{10} + i\,2\sin^2\frac{\pi}{10} = 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

For amplitude, $\tan\theta = \frac{\sin\frac{\pi}{10}}{\cos\frac{\pi}{10}} = \tan\frac{\pi}{10} \Rightarrow \theta = \frac{\pi}{10}$.

Example: 28 If $|z| = 4$ and *arg* $z = \frac{5\pi}{6}$, then $z =$ **[MP PET 1987]**

(a) $2\sqrt{3} - 2i$ (b) $2\sqrt{3} + 2i$ (c) $-2\sqrt{3} + 2i$ (d) $-\sqrt{3} + i$

Solution: [c] $|z| = 4$ and *arg* $z = \frac{5\pi}{6} = 150°$, Let $z = x + iy$, then $|z| = r = \sqrt{x^2 + y^2} = 4$ and $\theta = \frac{5\pi}{6} = 150°$

$\therefore\ x = r\cos\theta = 4\cos 150° = -2\sqrt{3}$ and $y = r\sin\theta = 4\sin 150° = 4\cdot\frac{1}{2} = 2.$

$\therefore\ z = x + iy = -2\sqrt{3} + 2i.$

Trick: Since $arg\ z = \frac{5\pi}{6} = 150°$, here the complex number must lie in second quadrant, so (a) and (b) rejected. Also $|z| = 4$, which satisfies (c) only.

Example: 29 If z and ω are to non-zero complex numbers such that $|z\omega| = 1$ and $arg\ (z) - arg\ (\omega) = \frac{\pi}{2}$, then $\bar{z}\omega$ is equal to

[AIEEE 2003]

(a) 1 (b) – 1 (c) i (d) – i

Solution: (d) $|z||\omega| = 1$ .....(i) and $arg\left(\frac{z}{\omega}\right) = \frac{\pi}{2} \Rightarrow \frac{z}{\omega} = i \Rightarrow \left|\frac{z}{\omega}\right| = 1$ .....(ii)

From equation (i) and (ii),

$|z| = |\omega| = 1$ and $\frac{z}{\omega} + \frac{\bar{z}}{\bar{\omega}} = 0;\ z\bar{\omega} + \bar{z}\omega = 0 \Rightarrow \bar{z}\omega = -z\bar{\omega} = \frac{-z}{\omega}\bar{\omega}\omega \Rightarrow \bar{z}\omega = -i|\omega|^2 = -i.$

2.8 Square Root of a Complex Number

Let $a + ib$ be a complex number such that $\sqrt{a + ib} = x + iy$, where x and y are real numbers. Then

$\sqrt{a + ib} = x + iy \Rightarrow a + ib = (x + iy)^2 \Rightarrow a + ib = (x^2 - y^2) + 2ixy$

$\Rightarrow x^2 - y^2 = a$ .....(i)

and $2xy = b$ .....(ii) [On equating real and imaginary parts]

Solving, $x = \pm\sqrt{\left(\frac{\sqrt{a^2 + b^2} + a}{2}\right)}$ and $y = \pm\sqrt{\left(\frac{\sqrt{a^2 + b^2} - a}{2}\right)}$

$\therefore\ \sqrt{a + ib} = \pm\left[\sqrt{\left(\frac{\sqrt{a^2 + b^2} + a}{2}\right)} - i\sqrt{\left(\frac{\sqrt{a^2 + b^2} - a}{2}\right)}\right]$

Therefore $\sqrt{a + ib} = \pm\left[\sqrt{\frac{|z| + a}{2}} + i\sqrt{\frac{|z| - a}{2}}\right]$ for $b>0$ $= \pm\left[\sqrt{\frac{|z| + a}{2}} - i\sqrt{\frac{|z| - a}{2}}\right]$ for $b<0$.

Note : ❑ To find the square root of $a - ib$, replace i by – i in the above results.

❑ The square root of i is $\pm\left(\frac{1 + i}{\sqrt{2}}\right)$, [Here b = 1]

❑ The square root of – i is $\pm\left(\frac{1 - i}{\sqrt{2}}\right)$, [Here b = –1]

Alternative method for finding the square root

(i) If the imaginary part is not even then multiply and divide the given complex number by 2. *e.g.* z = 8 – 15i here imaginary part is not even so write $z = \frac{1}{2}$ (16 – 30i) and let $a+ib = 16 - 30\,i$.

(ii) Now divide the numerical value of imaginary part of $a+ib$ by 2 and let quotient be P and find all possible two factors of the number P thus obtained and take that pair in which difference of squares of the numbers is equal to the real part of $a+ib$ *e.g.*, here numerical value of Im(16 – 30i) is 30. Now 30 = 2×15. All possible way to express 15 as a product of two are 1×15, 3×5 etc. here $5^2 - 3^2$ = 16 = *Re* (16– 30i) so we will take 5, 3.

(iii) Take i with the smaller or the greater factor according as the real part of $a + ib$ is positive or negative and if real part is zero then take equal factors of P and associate i with any one of them *e.g.*, Re(16 – 30i) > 0, we will take i with 3. Now complete the square and write down the square root of z.

e.g., $z = \frac{1}{2}[16-30i] = \frac{1}{2}\left[5^2 + (3i)^2 - 2\times 5\times 3i\right] = \frac{1}{2}[5-3i]^2 \Rightarrow \sqrt{z} = \pm\frac{1}{\sqrt{2}}(5-3i)$

Example: 30 The square root of $3-4i$ are

(a) $\pm(2-i)$ (b) $\pm(2+i)$ (c) $\pm(\sqrt{3}-2i)$ (d) $\pm(\sqrt{3}+2i)$

Solution: (a) $|z| = 5$, $\therefore \sqrt{3-4i} = \pm\left(\sqrt{\frac{5+3}{2}} - i\sqrt{\frac{5-3}{2}}\right) = \pm\,(2-i)$

Example: 31 $\sqrt{2i}$ equals **[Roorkee 1989]**

(a) $1+i$ (b) $1-i$ (c) $-\sqrt{2}i$ (d) None of these

Solution: (a) $z = 2i = a+bi \Rightarrow a = 0, b = 2, |z| = 2$

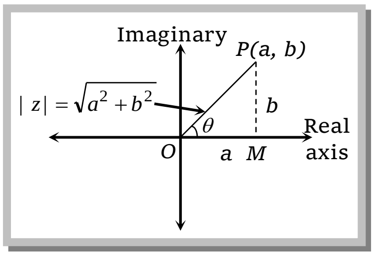
$\therefore \sqrt{z} = \pm\left(\sqrt{\frac{2+0}{2}} + i\sqrt{\frac{2-0}{2}}\right) = \pm(1+i)$

Trick: It is always better to square the options rather than finding the square root.

2.9 Representation of Complex Number

A complex number can be represented in the following from:

(1) **Geometrical representation (Cartesian representation):** The complex number $z = a+ib = (a,b)$ is represented by a point P whose coordinates are referred to rectangular axes XOX' and YOY' which are called real and imaginary axis respectively. Thus a complex number z is represented by a point in a plane, and corresponding to every point in this plane there exists a complex number such a plane is called argand plane or argand diagram or complex plane or gaussian



plane.

Note : ❑ Distance of any complex number from the origin is called the modules of complex number and is denoted by |z|, *i.e.*, $|z| = \sqrt{a^2 + b^2}$

❑ Angle of any complex number with positive direction of $x-$ axis is called amplitude or argument of z. *i.e.*, $amp\,(z) = arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$

(2) **Trigonometrical (Polar) representation :** In Δ OPM, let $OP = r$, then $a = r\cos\theta$ and $b = r\sin\theta$. Hence z can be expressed as $z = r(\cos\theta + i\sin\theta)$

where r = |z| and θ = principal value of argument of z.

For general values of the argument $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$

Note : ❑ Sometimes $(\cos\theta + i\sin\theta)$ is written in short as $cis\theta$.

(3) **Vector representation :** If P is the point (a, b) on the argand plane corresponding to the complex number $z = a + ib$.

Then $\overrightarrow{OP} = a\hat{i} + b\hat{j}$, $\therefore |\overrightarrow{OP}| = \sqrt{a^2 + b^2} = |z|$ and *arg z* = direction of the vector $\overrightarrow{OP} = \tan^{-1}\left(\frac{b}{a}\right)$

Therefore, complex number z can also be represented by $\overrightarrow{OP}$.

(4) **Eulerian representation (Exponential form) :** Since we have $e^{i\theta}$ = $\cos\theta + i\sin\theta$ and thus z can be expressed as $z = re^{i\theta}$, where $|z| = r$ and θ = *arg* (z)

Note : ❑ $e^{-i\theta} = (\cos\theta - i\sin\theta)$

❑ $e^{i\theta} + e^{-i\theta} = 2\cos\theta, e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

Example: 32 $\frac{1+7i}{(2-i)^2} =$ **[Roorkee 1998]**

(a) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (b) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (c) $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (d) None of these

Solution: (a) $\frac{1+7i}{(2-i)^2} = \frac{(1+7i)(3+4i)}{(3-4i)(3+4i)} = \frac{-25+25i}{25} = -1+i$

Let $z = x + iy = -1 + i$, $\therefore x = r\cos\theta = -1$ and $y = r\sin\theta = 1$

$\therefore \theta = \frac{3\pi}{4}$ and $r = \sqrt{2}$, Thus $\frac{1+7i}{(2-i)^2} = \sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

Alternative method: $\left|\frac{1+7i}{(2-i)^2}\right| = \left|\frac{1+7i}{3-4i}\right| = \sqrt{2}$ and *arg* $\left(\frac{1+7i}{3-4i}\right) = \tan^{-1}7 - \tan^{-1}\left(\frac{-4}{3}\right) = \tan^{-1}7 + \tan^{-1}\frac{4}{3} = \frac{3\pi}{4}$

$\therefore \frac{1+7i}{(2-i)^2} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Example: 33 If $-1 + \sqrt{-3} = re^{i\theta}$, then θ is equal to **[Rajasthan PET 1989; MP PMT 1999]**

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $-\frac{2\pi}{3}$

Solution: (c) Here $-1 + \sqrt{-3} = re^{i\theta} \Rightarrow -1 + i\sqrt{3} = re^{i\theta} = r\cos\theta + ir\sin\theta$

Equating real and imaginary part, we get $r\cos\theta = -1$ and $r\sin\theta = \sqrt{3}$

Hence $\tan\theta = -\sqrt{3} \Rightarrow \tan\theta = \tan\frac{2\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$.

Example: 34 Real part of $e^{e^{i\theta}}$ is **[Rajasthan PET 1995]**

(a) $e^{\cos\theta}[\cos(\sin\theta)]$ (b) $e^{\cos\theta}[\cos(\cos\theta)]$ (c) $e^{\sin\theta}[\sin(\cos\theta)]$ (d) $e^{\sin\theta}[\sin(\sin\theta)]$

Solution: (a) $e^{e^{i\theta}} = e^{(\cos\theta + i\sin\theta)} = e^{\cos\theta}.e^{i\sin\theta} = e^{\cos\theta}[\cos(i\sin\theta) + i\sin(\sin\theta)]$

$\therefore$ Real part of $e^{e^{i\theta}}$ is $e^{\cos\theta}[\cos(\sin\theta)]$.

Example: 35 If $\frac{1}{x} + x = 2\cos\theta$, then $x^n + \frac{1}{x^n}$ is equal to **[UPSEAT 2001]**

(a) $2\cos n\theta$ (b) $2\sin n\theta$ (c) $\cos n\theta$ (d) $\sin n\theta$

Solution: (a) Let $x = \cos\theta + i\sin\theta = e^{i\theta}$ then $x^n + \frac{1}{x^n} = e^{in\theta} + \frac{1}{e^{in\theta}} = e^{in\theta} + e^{-in\theta} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$.

2.10 Logarithm of a Complex Number

Let $z = x + iy$ and

$\log_e(x+iy) = a + ib$ .....(i)

$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ .....(ii)

then $x = r\cos\theta, y = r\sin\theta$, clearly $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

From equation (ii), $\log(x+iy) = \log_e(re^{i\theta}) = \log r + \log_e e^{i\theta} = \log_e r + i\theta = \log_e\sqrt{(x^2+y^2)} + i\tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{\log_e(z) = \log_e|z| + i\,amp\,z}$$

Obviously, the general value is $Log(z) = \log_e(z) + 2\pi ni \quad (-\pi < amp\,(z) < \pi)$

Example: 36 i^i is equal to **[EAMCET 1995]**

(a) $e^{\pi/2}$ (b) $e^{-\pi/2}$ (c) $-\pi/2$ (d) None of these

Solution: (b) Let $A = i^i$ then $\log A = \log i^i = i\log i \Rightarrow \log A = i\log(0+i) \Rightarrow \log A = i[\log 1 + i\pi/2]$ $(\because |i| = 1$ and $arg\,(i) = \pi/2)$

$\log A = i[0 + i\pi/2] = -\pi/2 \Rightarrow A = e^{-\pi/2}$.

Example: 37 $i\log\left(\frac{x-i}{x+i}\right)$ is equal to **[Rajasthan PET 2000]**

(a) $\pi + 2\tan^{-1}x$ (b) $\pi - 2\tan^{-1}x$ (c) $-\pi + 2\tan^{-1}x$ (d) $-\pi - 2\tan^{-1}x$

Solution: (b) Let $z = i\log\left(\frac{x-i}{x+i}\right) \Rightarrow \frac{z}{i} = \log\left(\frac{x-i}{x+i}\right) \Rightarrow \frac{z}{i} = \log\left[\frac{x-i}{x+i}\times\frac{x-i}{x-i}\right] = \log\left[\frac{x^2-1-2ix}{x^2+1}\right]$

$\Rightarrow \frac{z}{i} = \log\left[\frac{x^2-1}{x^2+1} - i\frac{2x}{x^2+1}\right]$ .....(i)

$\because \log(a+ib) = \log(re^{i\theta}) = \log r + i\theta = \log\sqrt{a^2+b^2} + i\tan^{-1}\left(\frac{b}{a}\right)$

Hence $\frac{z}{i} = \log\sqrt{\left(\frac{x^2-1}{x^2+1}\right)^2 + \left(\frac{-2x}{x^2+1}\right)^2} + i\tan^{-1}\left(\frac{-2x}{x^2-1}\right)$ (By equation (i))

$\frac{z}{i} = \log\frac{\sqrt{x^4+1-2x^2+4x^2}}{(x^2+1)^2} + i\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \log 1 + i(2\tan^{-1}x) = 0 + i(\tan^{-1}x)$

$\therefore z = i^2\,2\tan^{-1}x = -2\tan^{-1}x = \pi - 2\tan^{-1}x$.

2.11 Geometry of Complex Numbers

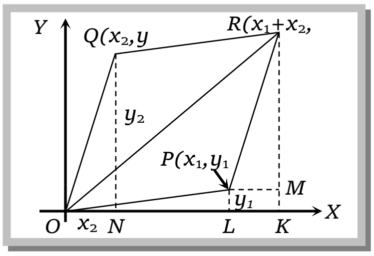
(1) **Geometrical representation of algebraic operations on complex numbers**

(i) **Sum:** Let the complex numbers $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$ be represented by the points P and Q on the argand plane.

Then sum of z_1 and z_2 *i.e.,* $z_1 + z_2$ is represented by the point R.

Complex number z can be represented by $\overrightarrow{OR}$.

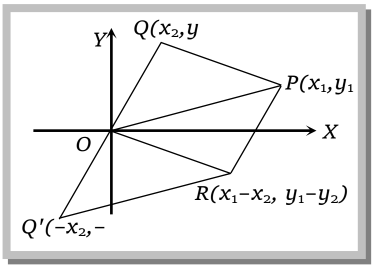
$= (x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = (z_1 + z_2) = (x_1, y_1) + (x_2, y_2)$



In vector notation, we have $z_1 + z_2 = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$

(ii) **Difference :** We first represent $-z_2$ by Q', so that QQ' is bisected at O.

The point R represents the difference $z_1 - z_2$.

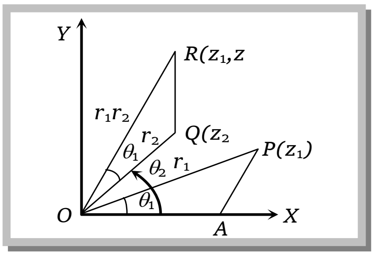


In parallelogram $ORPQ$, $\overrightarrow{OR} = \overrightarrow{QP}$

We have in vectorial notation $z_1 - z_2 = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{QO}$

$$= \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR} = \overrightarrow{QP}.$$

(iii) **Product :** Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$



$\therefore |z_1| = r_1$ and *arg* $(z_1) = \theta_1$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$\therefore |z_2| = r_2$ and *arg* $(z_2) = \theta_2$

Then, $z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$$

$\therefore$ $|z_1 z_2| = r_1 r_2$ and *arg* $(z_1 z_2) = \theta_1 + \theta_2$

R is the point representing product of complex numbers z_1 and z_2.

Important Tips

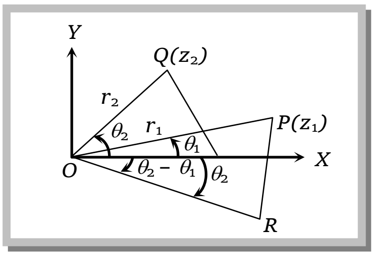
☞ *Multiplication of i : Since* $z = r(\cos\theta + i\sin\theta)$ *and* $i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ *then* $iz = \left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right]$

Hence, multiplication of z with i then vector for z rotates a right angle in the positive sense.

i.e., To multiply a vector by –1 is to turn it through two right angles.

i.e., To multiply a vector by $(\cos\theta + i\sin\theta)$ *is to turn it through the angle* θ *in the positive sense.*

(iv) **Division :** Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$



$\therefore |z_1| = r_1$ and *arg* $(z_1) = \theta_1$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$\therefore |z_2| = r_2$ and *arg* $(z_2) = \theta_2$

Then $\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}$ $\quad (z_2 \neq 0,\ r_2 \neq 0)$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}, arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

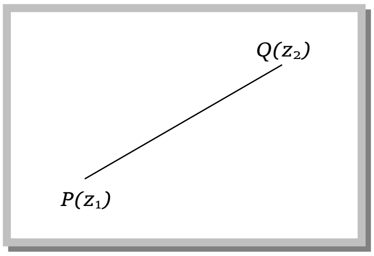
Note : ❑ The vertical angle R is $-(\theta_2 - \theta_1)$ *i.e.*, $\theta_1 - \theta_2$.

- If θ_1 and θ_2 are the principal values of z_1 and z_2 then $\theta_1+\theta_2$ and $\theta_1-\theta_2$ are not necessarily the principal value of $\arg(z_1 z_2)$ and $\arg(z_1/z_2)$.

2.12 Use of Complex Numbers in Co-ordinate Geometry

(1) **Distance formula :** The distance between two points $P(z_1)$ and $Q(z_2)$ is given by

$PQ=|z_2-z_1|$ = |*affix of Q – affix of P*|



Note : - The distance of point z from origin $|z-0|=|z|=|z-(0+i0)|$. Thus, modulus of a complex number z represented by a point in the argand plane is its distance from the origin.

- Three points $A(z_1), B(z_2)$ and $C(z_3)$ are collinear then $AB+BC=AC$

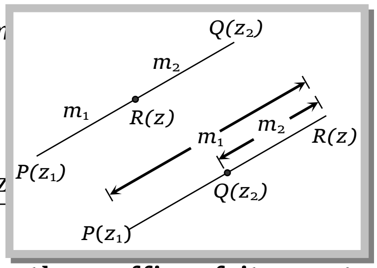
  i.e., $|z_1-z_2|+|z_2-z_3|=|z_1-z_3|$.

(2) **Section formula :** If $R(z)$ divides the joining of $P(z_1)$ and $Q(z_2)$ in the ratio $m_1:m_2(m_1,m_2>0)$

(i) If $R(z)$ divides the segment PQ internally in the ratio of $m_1:m_2$ then $z=\dfrac{m_1z_2+m_2z_1}{m_1+m_2}$

(ii) If $R(z)$ divides the segment PQ externally in the ratio of m

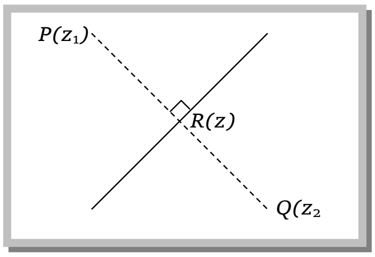
then $z=\dfrac{m_1z_2-m_2z_1}{m_1-m_2}$



Note : - If $R(z)$ is the mid point of PQ then affix of R is $\dfrac{z_1+z}{2}$

- If z_1,z_2,z_3 are affixes of the vertices of a triangle, then affix of its centroid is $\dfrac{z_1+z_2+z_3}{3}$.

(3) **Equation of the perpendicular bisector :** If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is moving point such that it is always at equal distance from $P(z_1)$ and



i.e., $PR=QR$ or $|z-z_1|=|z-z_2|$

$\Rightarrow |z-z_1|^2=|z-z_2|^2$

$\Rightarrow (z-z_1)\overline{(z-z_1)}=(z-z_2)\overline{(z-z_2)}$

$\Rightarrow (z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$

$\Rightarrow \bar{z}\ z(\bar{z}_1-\bar{z}_2)+\bar{z}(z_1-z_2)=z_1\bar{z}_1-z_2\bar{z}_2$ $\qquad \Rightarrow z(\bar{z}_1-\bar{z}_2)+\bar{z}(z_1-z_2)=|z_1|^2-|\bar{z}_2|^2$

Hence, z lies on the perpendicular bisector of z_1 and z_2.

(4) **Equation of a straight line**

(i) **Parametric form :** Equation of a straight line joining the point having affixes z_1 and z_2 is $z = tz_1 + (1-t)z_2$, when $t \in R$

(ii) **Non parametric form :** Equation of a straight line joining the points having affixes z_1 and z_2 is $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$.

Note : ❑ Three points z_1, z_2 and z_3 are collinear $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

(iii) **General equation of a straight line:** The general equation of a straight line is of the form $\bar{a}z + a\bar{z} + b = 0$, where a is complex number and b is real number.

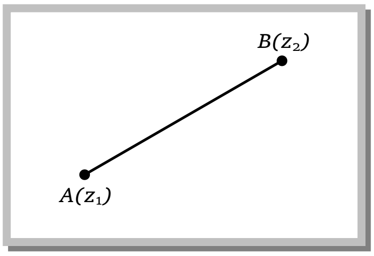
(iv) **Slope of a line :** The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{a}{\bar{a}} = -\frac{\text{coeff. of } \bar{z}}{\text{coeff. of } z}$ and real slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{\text{Re}(a)}{\text{Im}(a)} = -i\frac{(a+\bar{a})}{(a-\bar{a})}$.

Note : ❑ If α_1 and α_2 are the are the complex slopes of two lines on the argand plane, then

(i) If lines are perpendicular then $\alpha_1 + \alpha_2 = 0$ (ii) If lines are parallel then $\alpha_1 = \alpha_2$

❑ If lines $a\bar{z} + \bar{a}z + b = 0$ and $a_1\bar{z} + \bar{a}_1 z + b_1 = 0$ are the perpendicular or parallel, then $\left(\frac{-a}{\bar{a}}\right) + \left(\frac{-a_1}{\bar{a}_1}\right) = 0$ or $\frac{-a}{\bar{a}} = \frac{-a_1}{\bar{a}_1} \Rightarrow a\bar{a}_1 + a_1\bar{a} = 0$ or $a\bar{a}_1 - \bar{a}a_1 = 0$, where a, a_1 are the complex numbers and $b, b_1 \in R$.

(v) **Slope of the line segment joining two points :** If $A(z_1)$ and $B(z_2)$ represent two points in the argand plane then the complex slope of AB is defined by $\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

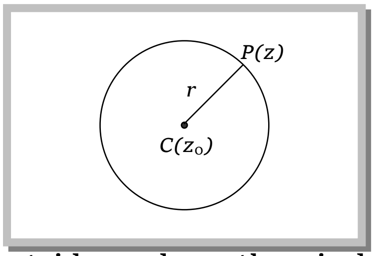


Note : ❑ If three points $A(z_1), B(z_2), C(z_3)$ are collinear then slope of AB = slope of BC = slope of AC

$$\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$$

(vi) **Length of perpendicular :** The length of perpendicular from a point z_1 to the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{|a| + |\bar{a}|}$ or $\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$

(5) **Equation of a circle :** The equation of a circle whose centre is at point having affix z_o and radius r is $|z - z_o| = r$



Note : ❑ If the centre of the circle is at origin and radius r, then its equation is $|z| = r$.

❑ $|z - z_0| < r$ represents interior of a circle $|z - z_0| = r$ and $|z - z_0| > r$ represent exterior of the circle $|z - z_o| = r$. Similarly, $|z - z_0| > r$ is the set of all points lying outside the circle and $|z - z_0| \geq r$ is the set of all points lying outside and on the circle $|z - z_0| = r$.

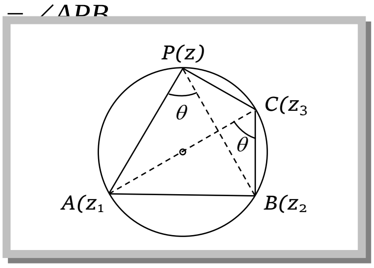
(i) **General equation of a circle :** The general equation of the circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ where a is complex number and $b \in R$.

$\therefore$ Centre and radius are – a and $\sqrt{|a|^2 - b}$ respectively.

Note : ❑ Rule to find the centre and radius of a circle whose equation is given:

- Make the coefficient of $z\bar{z}$ equal to 1 and right hand side equal to zero.
- The centre of circle will be = – a $= -\text{coefficent of } \bar{z}$
- Radius $= \sqrt{|a|^2 - \text{constant term}}$

(ii) **Equation of circle through three non-collinear points :** Let $A(z_1), B(z_2), C(z_3)$ are three points on the circle and $P(z)$ be any point on the circle, then $\angle ACB = \angle APB$



Using coni method

In ΔACB, $\dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{BC}{CA} e^{i\theta}$ .....(i)

In $\Delta\, APB$, $\dfrac{z_2 - z}{z_1 - z} = \dfrac{BP}{AP} e^{i\theta}$ .....(ii)

From (i) and (ii) we get

$\dfrac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)}$ = Real .....(iii)

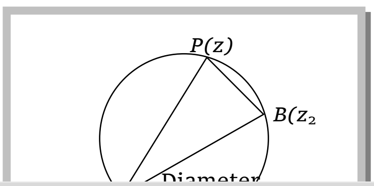
(iii) **Equation of circle in diametric form :** If end points of diameter represented by $A(z_1)$ and $B(z_2)$ and $P(z)$ be any point on circle then, $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

which is required equation of circle in diametric form.

(iv) **Other forms of circle :** (a) Equation of all circle which are orthogonal to $|z - z_1| = r_1$ and $|z - z_2| = r_2$. Let the circle be $|z - \alpha| = r$ cut given circles orthogonally

$\Rightarrow r^2 + r_1^2 = |\alpha - z_1|^2$ ......(i) and $r^2 + r_2^2 = |\alpha - z_2|^2$ .....(ii)

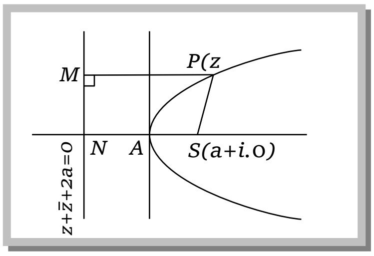
on solving $r_2^2 - r_1^2 = \alpha(\bar{z}_1 - \bar{z}_2) + \bar{\alpha}(z_1 - z_2) + |z_2|^2 - |z_1|^2$ and let $\alpha = a + ib$



(b) $\left|\frac{z-z_1}{z-z_2}\right| = k$ is a circle if $k \neq 1$ and a line if $k = 1$.

(c) The equation $|z-z_1|^2 + |z-z_2|^2 = k,$ will represent a circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$

(6) **Equation of parabola :** Now for parabola $SP = PM$



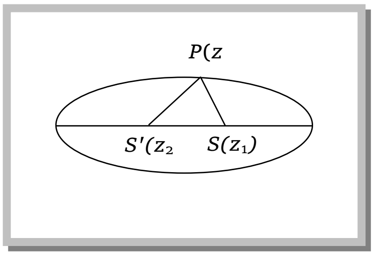
$$|z-a| = \frac{|z+\bar{z}+2a|}{2}$$

or $\quad z\bar{z} - 4a(z+\bar{z}) = \frac{1}{2}\{z^2 + (\bar{z})^2\}$

where $a \in R$ (focus)

Directrix is $z + \bar{z} + 2a = 0$

(7) **Equation of ellipse :** For ellipse $SP + S'P = 2a$

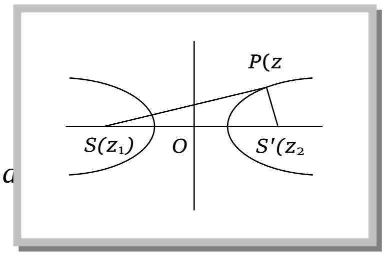


$\Rightarrow |z-z_1| + |z-z_2| = 2a$

where $2a \not> |z_1 - z_2|$ (since eccentricity <1)

Then point z describes an ellipse having foci at z_1 and z_2 and $a \in$

(8) **Equation of hyperbola :** For hyperbola $SP - S'P = 2a$



$\Rightarrow |z-z_1| - |z-z_2| = 2a$

where $2a \not< |z_1 - z_2|$ (since eccentricity >1)

Then point z describes a hyperbola having foci at z_1 and z_2 and a

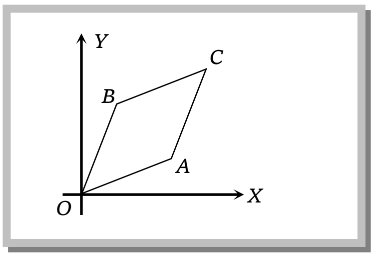
Example: 38 If in the adjoining diagram, A and B represent complex number z_1 and z_2 respectively, then C represents

(a) $z_1 + z_2$

(b) $z_1 - z_2$

(c) $z_1 . z_2$

(d) z_1 / z_2



Solution: (a) It is a fundamental concept.

Example: 39 If centre of a regular hexagon is at origin and one of the vertex on argand diagram is $1+2i,$ then its perimeter is

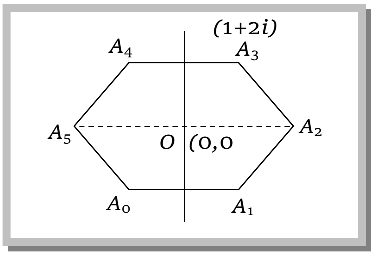
[Rajasthan PET 1999; Himachal CET 2002]

(a) $2\sqrt{5}$ (b) $6\sqrt{2}$ (c) $4\sqrt{5}$ (d) $6\sqrt{5}$

Solution: (d) Let the vertices be $z_0, z_1, \ldots\ldots, z_5$ w.r.t. centre O and $|z_0| = \sqrt{5}$

$\Rightarrow \quad A_0A_1 = |z_1 - z_0| = |z_0 e^{i\theta} - z_0| \quad = |z_0||\cos\theta + i\sin\theta - 1| \quad = \sqrt{5}\sqrt{(\cos\theta - 1)^2 + \sin^2\theta}$

$\Rightarrow \quad = \sqrt{5}\sqrt{2(1-\cos\theta)} \quad = \sqrt{5}.2\sin(\theta/2)$



$\Rightarrow A_0A_1 = \sqrt{5}.2\sin(\pi/6) = \sqrt{5} \quad \left(\because \theta = \frac{2\pi}{6} = \frac{\pi}{3}\right)$ .....(i)

Similarly, $A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5 = A_5A_0 = \sqrt{5}$

Hence, the perimeter of regular polygon is

$= A_0A_1 + A_1A_2 + A_2A_3 + A_3A_4 + A_4A_5 + A_5A_0 = 6\sqrt{5}$.

Example: 40 The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

[IIT Screening2001]

(a) Of area zero (b) Right-angled isosceles (c) Equilateral (d)

Solution: (b) Taking mod of both sides of given relation $\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|\frac{1}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$.

So, $|z_1 - z_3| = |z_2 - z_3|$. Also, $amp\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ or $amp\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \frac{\pi}{3}$ or $\angle z_2z_3z_1 = 60°$

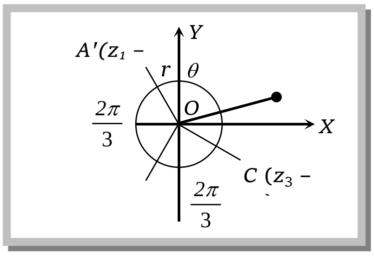
$\therefore$ The triangle has two sides equal and the angle between the equal sides $= 60°$. So it is equilateral.

Example: 41 Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2 =$

(a) z_0^2 (b) $-z_0^2$ (c) $3z_0^2$ (d) $-3z_0^2$

Solution: (c) Let r be the circum-radius of the equilateral triangle and ω the cube root of unity.

Let ABC be the equilateral triangle with z_1, z_2 and z_3 as its vertices A, B and C respectively with circumcentre $O'(z_0)$. The vectors $O'A, O'B, O'C$ are equal and parallel to OA', OB', OC' respectively.

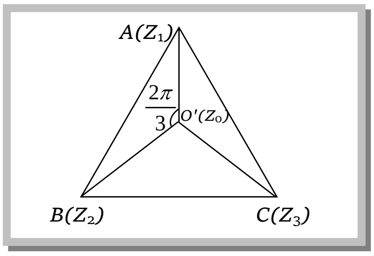


Then the vectors $\overrightarrow{OA'} = z_1 - z_0 = re^{i\theta}$

$\Rightarrow \overrightarrow{OB'} = z_2 - z_0 = re^{i(\theta + 2\pi/3)} = r\omega e^{i\theta}$

$\Rightarrow \overrightarrow{OC'} = z_3 - z_0 = re^{i(\theta + 4\pi/3)} = r\omega^2 e^{i\theta}$

$\therefore z_1 = z_0 + re^{i\theta}, z_2 = z_0 + r\omega e^{i\theta}, z_3 = z_0 + r\omega^2 e^{i\theta}$



Squaring and adding, we get,

$z_1^2 + z_2^2 + z_3^2 = 3z_0^2 + 2(1 + \omega + \omega^2)z_0 re^{i\theta} + (1 + \omega^2 + \omega^4)r^2 e^{i2\theta} = 3z_0^2$,

since $1 + \omega + \omega^2 = 0 = 1 + \omega^2 + \omega^4$.

Example: 42 The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order, if and only if

[IIT 1981,83]

(a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$ (c) $z_1 + z_2 = z_3 + z_4$ (d) None of these

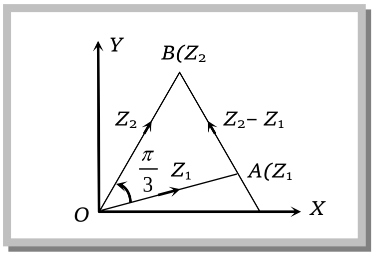
Solution: (b) Diagonals of a parallelogram $ABCD$ are bisected each other at a point *i.e.,* $\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$ $\Rightarrow$ $z_1 + z_3 = z_2 + z_4$.

Example: 43 If the complex number z_1, z_2 and the origin form an equilateral triangle then $z_1^2 + z_2^2 =$

(a) $z_1 z_2$ (b) $z_1 \bar{z}_2$ (c) $\bar{z}_2 z_1$ (d) $|z_1|^2 = |z_2|^2$

Solution: (a) Let *OA, OB* be the sides of an equilateral Δ *OAB* and *OA, OB* represent the complex numbers or vectors z_1, z_2 respectively.

From the equilateral Δ *OAB*, $\overrightarrow{AB} = Z_2 - Z_1$



$\therefore \ arg\left(\frac{z_2 - z_1}{z_2}\right) = arg(z_2 - z_1) - arg z_2 = \frac{\pi}{3}$ and $arg\left(\frac{z_2}{z_1}\right) = arg(z_2) - arg(z_1) = \frac{\pi}{3}$

Also, $\left|\frac{z_2 - z_1}{z_2}\right| = 1 = \left|\frac{z_2}{z_1}\right|$, since triangle is equilateral.

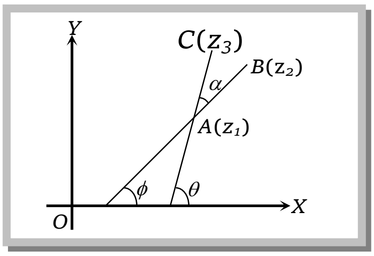
Thus the vectors $\frac{z_2 - z_1}{z_2}$ and $\frac{z_2}{z_1}$ have same modulus and same argument, which implies that the vectors are equal, that is $\frac{z_2 - z_1}{z_2} = \frac{z_2}{z_1} \Rightarrow z_1 z_2 - z_1^2 = z_2^2 \Rightarrow z_1^2 + z_2^2 = z_1 z_2$.

2.13 Rotation Theorem

Rotational theorem *i.e.*, angle between two intersecting lines. This is also known as coni method.

Let z_1, z_2 and z_3 be the affixes of three points *A, B* and *C* respectively taken on argand plane.

Then we have $\overrightarrow{AC} = z_3 - z_1$ and $\overrightarrow{AB} = z_2 - z_1$



and let $arg\ \overrightarrow{AC} = arg\ (z_3 - z_1) = \theta$ and $\overrightarrow{AB} = arg(z_2 - z_1) = \phi$

Let $\angle CAB = \alpha$, $\because \angle CAB = \alpha = \theta - \phi$

$= arg\ \overrightarrow{AC} - arg\ \overrightarrow{AB} = arg(z_3 - z_1) - arg(z_2 - z_1) = arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

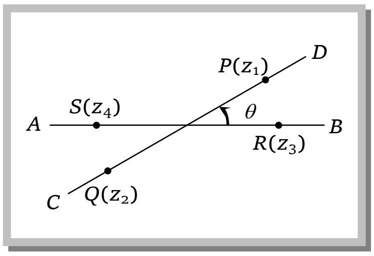
or angle between *AC* and *AB* = $arg\left(\frac{\text{affix of } C - \text{affix of } A}{\text{affix of } B - \text{affix of } A}\right)$

For any complex number *z* we have $z = |z|\, e^{i(arg z)}$

Similarly, $\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \left|\left(\frac{z_3 - z_1}{z_2 - z_1}\right)\right| e^{i\left(arg\frac{z_3 - z_1}{z_2 - z_1}\right)}$ or $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i(\angle CAB)} = \frac{AC}{AB} e^{i\alpha}$

Note : ❑ Here only principal values of the arguments are considered.

❑ $arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \theta$, if *AB* coincides with *CD*, then $arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = 0$ or $\pm\pi$, so that $\frac{z_1 - z_2}{z_3 - z_4}$ is real. It follows that if $\frac{z_1 - z_2}{z_3 - z_4}$ is real, then the points *A, B, C, D* are collinear.

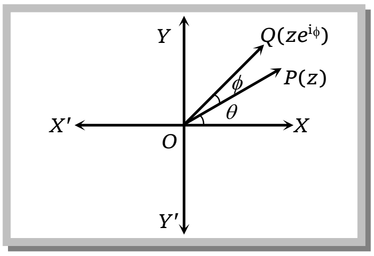


❑ If *AB* is perpendicular to *CD*, then *arg*

$\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\pi/2$, so $\frac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary. It follows that if $z_1 - z_2 = \pm k(z_3 - z_4)$, where k purely imaginary number, then AB and CD are perpendicular to each other.

(1) **Complex number as a rotating arrow in the argand plane :** Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ .....(i)

$r.e^{i\theta}$ be a complex number representing a point P in the argand plane.



Then $OP = |z| = r$ and $\angle POX = \theta$

Now consider complex number $z_1 = ze^{i\phi}$

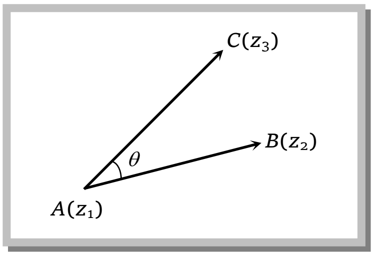
or $z_1 = re^{i\theta}.e^{i\phi} = re^{i(\theta+\phi)}$ {from (i)}

Clearly the complex number z_1 represents a point Q in the argand plane, when $OQ = r$ and $\angle QOX = \theta + \phi$.

Clearly multiplication of z with $e^{i\phi}$ rotates the vector $\overrightarrow{OP}$ through angle ϕ in anticlockwise sense. Similarly multiplication of z with $e^{-i\phi}$ will rotate the vector $\overrightarrow{OP}$ in clockwise sense.

Note : ❑ If z_1, z_2 and z_3 are the affixes of the points A, B and C such that $AC = AB$ and $\angle CAB = \theta$. Therefore, $\overrightarrow{AB} = z_2 - z_1$, $\overrightarrow{AC} = z_3 - z_1$.

Then $\overrightarrow{AC}$ will be obtained by rotating $\overrightarrow{AB}$ through an angle θ in anticlockwise sense, and therefore,



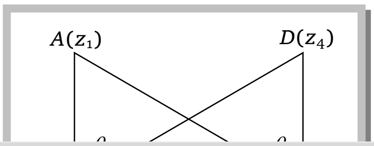
$\overrightarrow{AC} = \overrightarrow{AB}e^{i\theta}$ or $(z_3 - z_1) = (z_2 - z_1)e^{i\theta}$ or $\frac{z_3 - z_1}{z_2 - z_1} = e^{i\theta}$

❑ If A, B and C are three points in argand plane such that $AC = AB$ and $\angle CAB = \theta$ then use the rotation about A to find $e^{i\theta}$, but if $AC \neq AB$ use coni method.

❑ Let z_1 and z_2 be two complex numbers represented by point P and Q in the argand plane such that $\angle POQ = \theta$. Then, $z_1e^{i\theta}$ is a vector of magnitude $|z_1| = OP$ along $\overrightarrow{OQ}$ and $\frac{z_1e^{i\theta}}{|z_1|}$ is a unit vector along $\overrightarrow{OQ}$. Consequently, $|z_2|.\frac{z_1e^{i\theta}}{|z_1|}$ is a vector of magnitude $|z_2| = OQ$ along OQ i.e., $z_2 = \frac{|z_2|}{|z_1|}.z_1e^{i\theta} = z_2 = \left|\frac{z_2}{z_1}\right|$.

(2) **Condition for four points to be concyclic :** If points A, B, C and D are concyclic $\angle ABD = \angle ACD$

Using rotation theorem



In $\Delta ABD \dfrac{(z_1 - z_2)}{|z_1 - z_2|} = \dfrac{z_4 - z_2}{|z_4 - z_2|} e^{i\theta}$ .....(i)

In $\Delta ACD \dfrac{(z_1 - z_3)}{|z_1 - z_3|} = \dfrac{z_4 - z_3}{|z_4 - z_3|} e^{i\theta}$ .....(ii)

From (i) and (ii)

$$\frac{(z_1 - z_2)}{|z_1 - z_3|}\frac{(z_4 - z_3)}{|z_4 - z_2|} = \left|\frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}\right| = \text{Real}$$

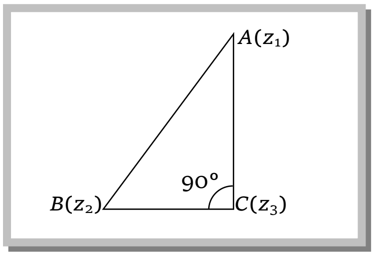
So if z_1, z_2, z_3 and z_4 are such that $\dfrac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}$ is real, then these four points are concyclic.

Example: 44 If complex numbers z_1, z_2 and z_3 represent the vertices *A, B* and *C* respectively of an isosceles triangle *ABC* of which $\angle C$ is right angle, then correct statement is

(a) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 z_3$ (b) $(z_3 - z_1)^2 = z_3 - z_2$

(c) $(z_1 - z_2)^2 = (z_1 - z_3)(z_3 - z_2)$ (d) $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

Solution: (d) $BC = AC$ and $\angle C = \pi/2$



By rotation about C in anticlockwise sense $CB = CAe^{i\pi/2}$

$\Rightarrow (z_2 - z_3) = (z_1 - z_3)e^{i\pi/2} = i(z_1 - z_3)$

$\Rightarrow (z_2 - z_3)^2 = -(z_1 - z_3)^2 \Rightarrow z_2^2 + z_3^2 - 2z_2 z_3 = -z_1^2 - z_3^2 + 2z_1 z_3$

$\Rightarrow z_1^2 + z_2^2 - 2z_1 z_2 = 2z_1 z_3 + 2z_2 z_3 - 2z_3^2 - 2z_1 z_2$

$\Rightarrow (z_1 - z_2)^2 = 2[(z_1 z_3 - z_3^2) - (z_1 z_2 - z_2 z_3)] \Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Example: 45 In the argand diagram, if *O,P* and *Q* represents respectively the origin, the complex numbers z and $z + iz$, then the angle $\angle OPQ$ is

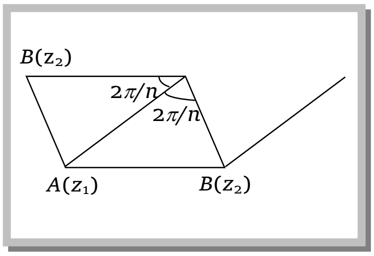
(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{2}$ (d) $\dfrac{2\pi}{3}$

Solution: (c) It is a fundamental concept.

Example: 46 The centre of a regular polygon of n sides is located at the point $z = 0$ and one of its vertex z_1 is known. If z_2 be the vertex adjacent to z_1, then z_2 is equal to

(a) $z_1\left(\cos\dfrac{2\pi}{n} \pm i\sin\dfrac{2\pi}{n}\right)$ (b) $z_1\left(\cos\dfrac{\pi}{n} \pm i\sin\dfrac{\pi}{n}\right)$ (c) $z_1\left(\cos\dfrac{\pi}{2n} \pm i\sin\dfrac{\pi}{2n}\right)$ (d) None of these

Solution: (a) Let A be the vertex with affix z_1. There are two possibilities of z_2 *i.e.*, z_2 can be obtained by rotating z_1 through $\dfrac{2\pi}{n}$ either in clockwise or in anticlockwise direction.



$\therefore \dfrac{z_2}{z_1} = \left|\dfrac{z_2}{z_1}\right| e^{\frac{i2\pi}{2}} \Rightarrow z_2 = z_1 e^{\frac{i2\pi}{2}}$ $(\because |z_2| = |z_1|)$

$\Rightarrow z_2 = z_1\left(\cos\dfrac{2\pi}{n} \pm i\sin\dfrac{2\pi}{n}\right)$

Example: 47 Let z_1, z_2, z_3 be three vertices of an equilateral triangle circumscribing the circle $|z| = \frac{1}{2}$. If $z_1 = \frac{1}{2} + \frac{\sqrt{3}i}{2}$ and z_1, z_2, z_3 are in anticlockwise sense then z_2 is

(a) $1+\sqrt{3}i$ (b) $1-\sqrt{3}\,i$ (c) 1 (d) – 1

Solution: (d) $z_2 = z_1 e^{i2\pi/3} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{-3}{4} - \frac{1}{4} = -1.$

2.14 Triangle Inequalities

In any triangle, sum of any two sides is greater than the third side and difference of any two side is less than the third side. By applying this basic concept to the set of complex numbers we are having the following results.

(1) $|z_1 + z_2| \le |z_1| + |z_2|$ (2) $|z_1 - z_2| \le |z_1| + |z_2|$

(3) $|z_1 + z_2| \ge ||z_1| - |z_2||$ (4) $|z_1 - z_2| \ge ||z_1| - |z_2||$

Note : ❑ In a complex plane $|z_1 - z_2|$ is the distance between the points z_1 and z_2.

❑ The equality $|z_1 + z_2| = |z_1| + |z_2|$ holds only when *arg* (z_1) = *arg* (z_2) i.e., z_1 and z_2 are parallel.

❑ The equality $|z_1 - z_2| = ||z_1| - |z_2||$ holds only when *arg* (z_1) – *arg* $(z_2) = \pi$ *i.e.*, z_1 and z_2 are antiparallel.

❑ In any parallelogram sum of the squares of its sides is equal to the sum of the squares of its diagonals *i.e.* $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

❑ Law of polygon *i.e.*, $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$

Important Tips

☞ *The area of the triangle whose vertices are z, iz and z + iz is* $\frac{1}{2}|z|^2$.

☞ *If* z_1, z_2, z_3 *be the vertices of a triangle then the area of the triangle is* $\frac{\sum (z_2 - z_3)|z_1|^2}{4iz_1}$.

☞ *Area of the triangle with vertices* z, wz *and* $z + wz$ *is* $\frac{\sqrt{3}}{4}|z^2|$.

☞ *If* z_1, z_2, z_3 *be the vertices of an equilateral triangle and* z_o *be the circumcentre, then* $z_1^2 + z_2^2 + z_3^2 + = 3z_0^2$.

☞ *If* $z_1, z_2, z_3 ..... z_n$ *be the vertices of a regular polygon of n sides and* z_0 *be its centroid, then* $z_1^2 + z_2^2 + ... + z_n^2 = nz_0^2$.

☞ *If* z_1, z_2, z_3 *be the vertices of a triangle, then the triangle is equilateral iff* $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$ *or* $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ *or* $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$.

☞ *If* $z_1, z_2 z_3$ *are the vertices of an isosceles triangle, right angled at* z_2 *then* $z_1^2 + z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

☞ *If* z_1, z_2, z_3 *are the vertices of right-angled isosceles triangle, then* $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

☞ *If one of the vertices of the triangle is at the origin i.e.,* $z_3 = 0$, *then the triangle is equilateral iff* $z_1^2 + z_2^2 - z_1 z_2 = 0$.

☞ *If* z_1, z_2, z_3 *and* z_1', z_2', z_3' *are the vertices of a similar triangle, then* $\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0$.

☞ *If* z_1, z_2, z_3 *be the affixes of the vertices A, B, C respectively of a triangle ABC, then its orthocentre is* $\frac{a(\sec A)z_1 + b(\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C}$.

Example: 48 The points $1+3i, 5+i$ and $3+2i$ in the complex plane are

(a) Vertices of a right angled triangle (b) Collinear

(c) Vertices of an obtuse angled triangle (d) Vertices of an equilateral triangle

Solution: (b) Let $z_1 = 1+3i, z_2 = 5+i$ and $z_3 = 3+2i$. Then area of triangle $A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$, Hence z_1, z_2 and z_3 are collinear.

Example: 49 If $z = x + iy$, then area of the triangle whose vertices are points z, iz and $z + iz$ is

[IIT 1986; MP PET 1997, 2001; DCE 1997; AMU 2000; UPSEAT 2002]

(a) $2|z|^2$ (b) $\frac{1}{2}|z|^2$ (c) $|z|^2$ (d) $\frac{3}{2}|z|^2$

Solution: (b) Let $z = x+iy$, $z + iz = (x-y) + i(x+y)$ and $iz = -y + ix$

If A denotes the area of the triangle formed by $z, z+iz$ and iz, then

$$A = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x-y & x+y & 1 \\ -y & x & 1 \end{vmatrix}$$ (Applying transformation $R_2 \to R_2 - R_1 - R_3$)

We get $A = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 0 & 0 & -1 \\ -y & x & 0 \end{vmatrix} = \frac{1}{2}(x^2 + y^2) = \frac{1}{2}|z|^2$.

Example: 50 $|z_1 + z_2| = |z_1| + |z_2|$ is possible if **[MP PET 1999]**

(a) $z_2 = \bar{z}_1$ (b) $z_2 = \frac{1}{z_1}$ (c) $\arg(z_1) = \arg(z_2)$ (d) $|z_1| = |z_2|$

Solution: (c) Squaring both sides, we get

$|z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow 2|z_1||z_2|\cos(\theta_1 - \theta_2) = 2|z_1||z_2| \Rightarrow \cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0^o \Rightarrow \theta_1 = \theta_2$

Hence *arg* $(z_1) =$ *arg* (z_2)

Trick: Let z_1 and z_2 are the two sides of a triangle. By applying triangle inequality $(z_1 + z_2)$ is the third side. Equality holds only when $\theta_1 = \theta_2$ *i.e.*, z_1 and z_2 are parallel.

2.15 Standard Loci in the Argand Plane

(1) If z is a variable point in the argand plane such that *arg* $(z) = \theta$, then locus of z is a straight line (excluding origin) through the origin inclined at an angle θ with x–axis.

(2) If z is a variable point and z_1 is a fixed point in the argand plane such that *arg* $(z - z_1) = \theta$, then locus of z is a straight line passing through the point representing z_1 and inclined at an angle θ with x-axis. Note that the point z_1 is excluded from the locus.

(3) If z is a variable point and z_1, z_2 are two fixed points in the argand plane, then

(i) $|z - z_1| = |z - z_2|$ $\Rightarrow$ Locus of z is the perpendicular bisector of the line segment joining z_1 and z_2

(ii) $|z - z_1| + |z - z_2|$ = constant $(\neq |z_1 - z_2|)$ $\Rightarrow$ Locus of z is an ellipse

(iii) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ $\Rightarrow$ Locus of z is the line segment joining z_1 and z_2

(iv) $|z - z_1| - |z - z_2| = |z_1 - z_2|$ $\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2.

(v) $|z - z_1| - |z - z_2| = \text{constant} \quad (\neq |z_1 - z_2|)$ $\Rightarrow$ Locus of z is a hyperbola.

(vi) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|$ $\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the extremities of diameter.

(vii) $|z - z_1| = k|z - z_2| \quad k \neq 1$ $\Rightarrow$ Locus of z is a circle.

(viii) $arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ (fixed) $\Rightarrow$ Locus of z is a segment of circle.

(ix) $arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\pi/2$ $\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the vertices of diameter.

(x) $arg\left(\frac{z - z_1}{z - z_2}\right) = 0$ or π $\Rightarrow$ Locus z is a straight line passing through z_1 and z_2.

(xi) The equation of the line joining complex numbers z_1 and z_2 is given by $\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

or $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

Example: 51 The locus of the points z which satisfy the condition $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$ is **[Rajasthan PET 2000,2002; MP PET 20**

(a) A straight line (b) A circle (c) A parabola (d) None of these

Solution: (c) We have $\frac{z-1}{z+1} = \frac{x + iy - 1}{x + iy + 1} = \frac{(x^2 + y^2 - 1) + 2iy}{(x+1)^2 + y^2}$

$\Rightarrow arg\frac{z-1}{z+1} = \tan^{-1}\frac{2y}{x^2 + y^2 - 1}$

Hence $\tan^{-1}\frac{2y}{x^2 + y^2 - 1} = \frac{\pi}{3}$

$\Rightarrow \quad \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{3} = \sqrt{3} \Rightarrow x^2+y^2-1 = \frac{2}{\sqrt{3}}y \Rightarrow x^2+y^2-\frac{2}{\sqrt{3}}y-1=0$, which is obviously a circle.

Example: 52 If $|z^2-1| = |z|^2+1$, then z lies on **[AIEEE 2004]**

(a) An ellipse (b) The imaginary axis (c) A circle (d) The real axis

Solution: (b) $|z^2-1| = |z|^2+1$

$\Rightarrow |z-1|^2 |z+1|^2 = (z\bar{z}+1)^2 \Rightarrow (z-1)(\bar{z}-1)(z+1)(\bar{z}+1) = (z\bar{z}+1)^2 \Rightarrow z+\bar{z}=0$

$\therefore$ z lies on imaginary axis.

Example: 53 The locus of the point z satisfying $arg\left(\frac{z-1}{z+1}\right) = k$. (where k is non-zero) is

(a) Circle with centre on y–axis (b) Circle with centre on x–axis

(b) A straight line parallel to x–axis (d) A straight line making an angle 60° with x–axis

Solution: (a) $arg\left(\frac{z-1}{z+1}\right) = k \Rightarrow arg\left[\frac{(x-1)+iy}{(x+1)+iy}\right] = k \Rightarrow arg[(x-1)+iy] - arg[(x+1)+iy] = k$

$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = k \Rightarrow \tan^{-1}\left[\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\frac{y^2}{x^2-1}}\right] = k \Rightarrow \tan k = \frac{y(x+1)-y(x-1)}{x^2+y^2-1} = \frac{2y}{x^2+y^2-1}$

$\Rightarrow \frac{2y}{\tan k} = x^2+y^2-1 \Rightarrow x^2+y^2-\frac{2y}{\tan k}-1=0$

It is an equation of circle whose centre is $(-g,-f) = (0, \cot k)$ on y–axis.

Example: 54 The locus of z satisfying the inequality $\log_{1/3}|z+1| > \log_{1/3}|z-1|$ is

(a) $R(z) < 0$ (b) $R(z) > 0$ (c) $I(z) < 0$ (d) None of these

Solution: (a) $\log_{1/3}|z+1| > \log_{1/3}|z-1|$

$\Rightarrow |z+1| < |z-1| \Rightarrow x^2+1+2x+y^2 < x^2+1-2x+y^2 \Rightarrow x<0 \Rightarrow \text{Re}(z)<0.$

Example: 55 If $\alpha+i\beta = \tan^{-1}(z), z = x+iy$ and α is constant, the locus of 'z' is **[EAMCET 1995; KCET 1996]**

(a) $x^2+y^2+2x\cot 2\alpha = 1$ (b) $\cot 2\alpha(x^2+y^2) = 1+x$ (c) $x^2+y^2+2y\tan 2\alpha = 1$ (d) $x^2+y^2+2x\sin 2\alpha = 1$

Solution: (a) $\tan(\alpha+i\beta) = x+iy$

$\therefore \tan(\alpha-i\beta) = x-iy$ (conjugate), α is a constant and β is known to be eliminated

$\tan 2\alpha = \tan(\overline{\alpha+i\beta}+\overline{\alpha-i\beta}) \Rightarrow \tan 2\alpha = \frac{x+iy+x-iy}{1-(x^2+y^2)} \Rightarrow 1-(x^2+y^2) = 2x\cot 2\alpha$

$\therefore x^2+y^2+2x\cot 2\alpha = 1.$

2.16 De' Moivre's Theorem

(1) If n is any rational number, then $(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(2) If $z = (\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)(\cos\theta_3+i\sin\theta_3).....(\cos\theta_n+i\sin\theta_n)$

then $z = \cos(\theta_1+\theta_2+\theta_3+.....+\theta_n) + i\sin(\theta_1+\theta_2+\theta_3+.....+\theta_n)$, where $\theta_1, \theta_2, \theta_3.....\theta_n \in R$.

(3) If $z = r(\cos\theta+i\sin\theta)$ and n is a positive integer, then $z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right)+i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$,

where $k = 0, 1, 2, 3, ...(n-1)$.

(4) If $p, q \in z$ and $q \neq 0$, then $(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$,

where $k = 0, 1, 2, 3.....(q-1)$.

Deductions: If $n \in Q$, then

(i) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

(ii) $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

(iii) $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

(iv) $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$

Applications

(i) In finding the expansions of trigonometric functions *i.e.* $\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - ......$

$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + {}^nC_5 \cos^{n-5}\theta \sin^5\theta - ....$

(ii) In finding the roots of complex numbers.

(iii) In finding the complex solution of algebraic equations.

Note : ❑ This theorem is not valid when *n* is not a rational number or the complex number is not in the form of $\cos\theta + i\sin\theta$.

Powers of complex numbers : Let $z = x + iy = r(\cos\theta + i\sin\theta)$

$\therefore\ z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$

Number	*x + iy* form	Standard complex form	General
1	$1 + i0$	$\cos 0 + i\sin 0$	$\cos 2n\pi + i\sin 2n\pi$
– 1	$-1 + i0$	$\cos\pi + i\sin\pi$	$\cos(2n+1)\pi + i\sin(2n+1)\pi$
i	$0 + i(1)$	$\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$	$\cos(4n+1)\frac{\pi}{2} + i\sin(4n+1)\frac{\pi}{2}$
$-i$	$0 + i(-1)$	$\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$	$\cos(4n+1)\frac{\pi}{2} - i\sin(4n+1)\frac{\pi}{2}$

Example: 56 If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then

(a) $a = 2, b = -1$ (b) $a = 1, b = 0$ (c) $a = 0, b = 1$ (d) $a = -1, b = 2$

Solution: (b) $\frac{1-i}{1+i} \times \frac{1-i}{1-i} = -i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) \Rightarrow (-i)^{100} = \cos(-50\pi) + i\sin(-50\pi) = 1 + i(0) \Rightarrow a = 1,\ b = 0$

Example: 57 If $x_r = \cos\left(\frac{\pi}{2^r}\right) + i\sin\left(\frac{\pi}{2^r}\right)$, then $x_1.x_2.x_3..........\infty$ is

[Rajasthan PET 1990, 2000; Karnataka CET 2000; UPSEAT 1990; Haryana CEE 1998; BIT Ranchi 1996]

(a) –3 (b) –2 (c) –1 (d) 0

Solution: (c) $x_1.x_2.x_3.....$ upto $\infty = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)..................$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + ...\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + ...\right) = \cos\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right) + i\sin\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right) = \cos\pi + i\sin\pi = -1$$

Example: 58 If $z_r = \cos\frac{r\alpha}{n^2} + i\sin\frac{r\alpha}{n^2}$, where $r = 1,2,3,......, n$, then $\lim_{n\to\infty} z_1 z_2 z_3 ..... z_n$ is equal to **[UPSEAT 2001]**

(a) $\cos\alpha + i\sin\alpha$ (b) $\cos(\alpha/2) - i\sin(\alpha/2)$ (c) $e^{i\alpha/2}$ (d) $\sqrt[3]{e^{i\alpha}}$

Solution: (c) $z_r = \cos\frac{r\alpha}{n^2} + i\sin\frac{r\alpha}{n^2} \Rightarrow z_1 = \cos\frac{\alpha}{n^2} + i\sin\frac{\alpha}{n^2}$;

$z_2 = \cos\frac{2\alpha}{n^2} + i\sin\frac{2\alpha}{n^2}$;..................

$$\Rightarrow z_n = \cos\frac{n\alpha}{n^2} + i\sin\frac{n\alpha}{n^2} \Rightarrow \lim_{n\to\infty}(z_1, z_2, z_3, ...z_n) = \lim_{n\to\infty}\left[\cos\left\{\frac{\alpha}{n^2}(1+2+3+.....+n)\right\} + i\sin\left\{\frac{\alpha}{n^2}(1+2+3+.....+n)\right\}\right]$$

$$= \lim_{n\to\infty}\left[\cos\frac{\alpha\, n(n+1)}{2n^2} + i\sin\frac{\alpha\, n(n+1)}{2n^2}\right] = \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2} = e^{i\alpha/2}.$$

Example: 59 $\left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n =$ **[Kerala (Engg.) 2002]**

(a) $\cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$ (b) $\cos\left(\frac{n\pi}{2} + n\theta\right) + i\sin\left(\frac{n\pi}{2} + n\theta\right)$

(c) $\sin\left(\frac{n\pi}{2} - n\theta\right) + i\cos\left(\frac{n\pi}{2} - n\theta\right)$ (d) $\cos n\left(\frac{n\pi}{2} + n\theta\right) + i\sin n\left(\frac{n\pi}{2} + n\theta\right)$

Solution: (a) $\left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \left(\frac{1+\cos\alpha + i\sin\alpha}{1+\cos\alpha - i\sin\alpha}\right)^n$ $\left(\text{where } \alpha = \frac{\pi}{2} - \theta\right)$

$$= \left(\frac{2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2} - 2i\sin\frac{\alpha}{2}.\cos\frac{\alpha}{2}}\right)^n = \left(\frac{\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}}\right)^n = \left[\frac{cis\left(\frac{\alpha}{2}\right)}{cis\left(-\frac{\alpha}{2}\right)}\right]^n = \left\{cis\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right)\right\}^n = cis(n\alpha)$$

$$= cis\, n\left(\frac{\pi}{2} - \theta\right) = cis\left(\frac{n\pi}{2} - \theta\right) = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right).$$

2.17 Roots of a Complex Number

(1) **n^{th} roots of complex number** ($z^{1/n}$) **:** Let $z = r(\cos + i\sin\theta)$ be a complex number. To find the roots of a complex number, first we express it in polar form with the general value of its

amplitude and use the De' moivre's theorem. By using De'moivre's theorem n^{th} roots having n distinct values of such a complex number are given by

$$z^{1/n} = r^{1/n}\left[\cos\frac{2m\pi+\theta}{n} + i\sin\frac{2m\pi+\theta}{n}\right], \text{where } m = 0,1,2,.....,(n-1).$$

Properties of the roots of $z^{1/n}$:

(i) All roots of $z^{1/n}$ are in geometrical progression with common ratio $e^{2\pi i/n}$.

(ii) Sum of all roots of $z^{1/n}$ is always equal to zero.

(iii) Product of all roots of $z^{1/n} = (-1)^{n-1}z$.

(iv) Modulus of all roots of $z^{1/n}$ are equal and each equal to $r^{1/n}$ or $|z|^{1/n}$.

(v) Amplitude of all the roots of $z^{1/n}$ are in A.P. with common difference $\frac{2\pi}{n}$.

(vi) All roots of $z^{1/n}$ lies on the circumference of a circle whose centre is origin and radius equal to $|z|^{1/n}$. Also these roots divides the circle into n equal parts and forms a polygon of n sides.

(2) **The n^{th} roots of unity :** The n^{th} roots of unity are given by the solution set of the equation

$$x^n = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$x = [\cos 2k\pi + i\sin 2k\pi]^{1/n}$$

$$x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \text{ where } k = 0,1,2,.....,(n-1).$$

Properties of n^{th} roots of unity

(i) Let $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i(2\pi/n)}$, the n^{th} roots of unity can be expressed in the form of a series *i.e.,* $1,\alpha,\alpha^2,.....\alpha^{n-1}$. Clearly the series is G.P. with common difference α *i.e.,* $e^{i(2\pi/n)}$.

(ii) The sum of all n roots of unity is zero *i.e.,* $1+\alpha+\alpha^2+.....+\alpha^{n-1} = 0$.

(iii) Product of all n roots of unity is $(-1)^{n-1}$.

(iv) Sum of p^{th} power of n roots of unity

$$1+\alpha^p+\alpha^{2p}+.....+\alpha^{(n-1)p} = \begin{cases} 0, \text{when } p \text{ is not multiple of } n \\ n, \text{when } p \text{ is a multiple of } n \end{cases}$$

(v) The n, n^{th} roots of unity if represented on a complex plane locate their positions at the vertices of a regular plane polygon of n sides inscribed in a unit circle having centre at origin, one vertex on positive real axis.

Note : ❑ $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + ... + x + 1)$

❑ $(\sin\theta + i\cos\theta) = -i^2\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta)$

(3) **Cube roots of unity :** Cube roots of unity are the solution set of the equation $x^3 - 1 = 0 \Rightarrow$ $x = (1)^{1/3} \Rightarrow x = (\cos 0 + i\sin 0)^{1/3} \Rightarrow x = \cos\frac{2k\pi}{3} + i\sin\left(\frac{2k\pi}{3}\right)$, where $k = 0,1,2$

Therefore roots are $1, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ or $1, e^{2\pi i/3}, e^{4\pi i/3}$.

Alternative : $x = (1)^{1/3} \Rightarrow x^3 - 1 = 0 \Rightarrow (x-1)(x^2 + x + 1) = 0$

$$x = 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$$

If one of the complex roots is ω, then other root will be ω^2 or vice-versa.

Properties of cube roots of unity

(i) $1 + \omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

(iii) $1 + \omega^r + \omega^{2r} = \begin{cases} 0, \text{ if } r \text{ not a multiple of } 3 \\ 3, \text{ if } r \text{ is a multiple of } 3 \end{cases}$

(iv) $\overline{\omega} = \omega^2$ and $(\overline{\omega})^2 = \omega$ and $\omega . \overline{\omega} = \omega^3$.

(v) Cube roots of unity from a G.P.

(vi) Imaginary cube roots of unity are square of each other *i.e.,* $(\omega)^2 = \omega^2$ and $(\omega^2)^2 = \omega^3.\omega = \omega$.

(vii) Imaginary cube roots of unity are reciprocal to each other *i.e.,* $\frac{1}{\omega} = \omega^2$ and $\frac{1}{\omega^2} = \omega$.

(viii) The cube roots of unity by, when represented on complex plane, lie on vertices of an equilateral triangle inscribed in a unit circle having centre at origin, one vertex being on positive real axis.

(ix) A complex number $a + ib$, for which $|a : b| = 1 : \sqrt{3}$ or $\sqrt{3} : 1$, can always be expressed in terms of i, ω, ω^2.

Note : ❑ If $\omega = \frac{-1 + i\sqrt{3}}{2} = e^{2\pi i/3}$, then $\omega^2 = \frac{-1 - i\sqrt{3}}{2} = e^{-4\pi i/3} = e^{-2\pi i/3}$ or vice-versa $\omega . \overline{\omega} = \omega^3$.

❑ $a + b\omega + c\omega^2 = 0 \Rightarrow a = b = c$, if a, b, c are real.

❑ Cube root of – 1 are $-1, -\omega, -\omega^2$.

Important Tips

☞ $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

☞ $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

☞ $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

☞ $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$

☞ $x^2 + y^2 = (x + iy)(x - iy)$

☞ $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

☞ $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

☞ $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

☞ $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

Fourth roots of unity : The four, fourth roots of unity are given by the solution set of the equation $x^4 - 1 = 0$. $\Rightarrow (x^2-1)(x^2+1)=0 \Rightarrow x = \pm 1, \pm i$

Note : ❑ Sum of roots = 0 and product of roots =–1.

❑ Fourth roots of unity are vertices of a square which lies on coordinate axes.

Continued product of the roots

If $z = r(\cos\theta + i\sin\theta)$ *i.e.*, $|z| = r$ and $amp\,(z) = \theta$ then continued product of roots of $z^{1/n}$ is

$= r(\cos\phi + i\sin\phi)$, where $\phi = \sum_{m=0}^{n-1} \frac{2m\pi + \theta}{n} = (n-1)\pi + \theta$.

Thus continued product of roots of $z^{1/n} = r[\cos\{(n-1)\pi + \theta\} + i\sin\{(n-1)\pi + \theta\}] = \begin{cases} z, \text{ if } n \text{ is odd} \\ -z, \text{ if } n \text{ is even} \end{cases}$

Similarly, the continued product of values of $z^{m/n}$ is $= \begin{cases} z^m, & \text{if } n \text{ is odd} \\ (-z)^m, & \text{if } n \text{ is even} \end{cases}$

Important Tips

☞ *If* $x + \frac{1}{x} = 2\cos\theta$ *or* $x - \frac{1}{x} = 2i\sin\theta$ *then* $x = \cos\theta + i\sin\theta, \frac{1}{x} = \cos\theta - i\sin\theta, x^n + \frac{1}{x^n} = 2\cos n\theta, x^n - \frac{1}{x^n} = 2i\sin n\theta$.

☞ *If n be a positive integer then ,* $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$.

☞ *If z is a complex number, then* e^z *is periodic.*

☞ n^{th} *root of –1 are the solution of the equation* $z^n + 1 = 0$

$z^n - 1 = (z-1)(z-\alpha)(z-\alpha^2).....(z-\alpha^{n-1})$, *where* $\alpha = n^{th}$ *root of unity*

$z^n - 1 = (z-1)(z+1)\prod_{r=1}^{(n-2)/2}\left(z^2 - 2z\cos\frac{2r\pi}{n} + 1\right)$, *if n is even.*

$$z^n + 1 = \begin{cases} \prod_{r=0}^{(n-2)/2}\left[z^2 - 2z\cos\left(\frac{(2r+1)\pi}{n}\right) + 1\right], \text{ if } n \text{ is even.} \\ (z+1)\prod_{r=0}^{(n-3)/2}\left[z^2 - 2z\cos\left(\frac{(2r+1)\pi}{n}\right) + 1\right], \text{ if } n \text{ is odd.} \end{cases}$$

☞ *If* $x = \cos\alpha + i\sin\alpha, y = \cos\beta + i\sin\beta, z = \cos\gamma + i\sin\gamma$ *and given,* $x + y + z = 0$, *then*

(i) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ *(ii)* $yz + zx + xy = 0$ *(iii)* $x^2 + y^2 + z^2 = 0$ *(iv)* $x^3 + y^3 + z^3 = 3xyz$

then, putting, values if x, y, z *in these results*

$x + y + z = 0 \Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma \Rightarrow yz + zx + xy = 0 \Rightarrow \begin{cases} \cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0 \\ \sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0 \end{cases}$

$x^2 + y^2 + z^2 = 0 \Rightarrow \begin{cases} \sum \cos 2\alpha = 0 \\ \sum \sin 2\alpha = 0, \end{cases}$ *the summation consists 3 terms*

$x^3 + y^3 + z^3 = 3xyz$, *gives similarly*

$\sum \cos 3\alpha = 3\cos(\alpha+\beta+\gamma) \Rightarrow \sum \sin 3\alpha = 3\sin(\alpha+\beta+\gamma)$

If the condition given be $x+y+z=xyz$, *then* $\sum \cos\alpha = \cos(\alpha+\beta+\gamma)$ *etc.*

Example: 60 If the cube roots of unity be $1, \omega, \omega^2$, then the roots of the equation $(x-1)^3+8=0$ are **[DCE 2000; IIT 1979; UPSEAT 1986]**

(a) $-1, 1+2\omega, 1+2\omega^2$ (b) $-1, 1-2\omega, 1-2\omega^2$ (c) $-1, 1, -1$ (d) None of these

Solution: (c) $(x-1)^3=-8 \Rightarrow x-1=(-8)^{1/3} \Rightarrow x-1=-2,-2\omega,-2\omega^2 \Rightarrow x=-1, 1-2\omega, 1-2\omega^2$

Example: 61 ω is an imaginary cube root of unity. If $(1+\omega^2)^m=(1+\omega^4)^m$, then least positive integral value of m is **[IIT Screening 2004]**

(a) 6 (b) 5 (c) 4 (d) 3

Solution: (d) The given equation reduces to $(-\omega)^m=(-\omega^2)^m \Rightarrow \omega^m=1 \Rightarrow m=3$.

Example: 62 If ω is the cube root of unity, then $(3+5\omega+3\omega^2)^2+(3+3\omega+5\omega^2)^2=$ **[MP PET 1999]**

(a) 4 (b) 0 (c) – 4 (d) None of these

Solution: (c) $(3+5\omega+3\omega^2)^2+(3+3\omega+5\omega^2)^2=(3+3\omega+3\omega^2+2\omega)^2+(3+3\omega+3\omega^2+2\omega^2)^2$

$=(2\omega)^2+(2\omega^2)^2=4\omega^2+4\omega^4=4(-1)=-4$ $(\because 1+\omega+\omega^2=0, \omega^3=1)$

Example: 63 If $i=\sqrt{-1}$, then $4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to **[IIT 1999]**

(a) $1-i\sqrt{3}$ (b) $-1+i\sqrt{3}$ (c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

Solution: (c) Given equation is $4+5\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{365}$

$=4+5\omega^{334}+3\omega^{365}=4+5\omega+3\omega^2=1+2\omega=1+2\left(\frac{-1+i\sqrt{3}}{2}\right)=i\sqrt{3}$

Example: 64 Let ω is an imaginary cube root of unity then the value of

$2(\omega+1)(\omega^2+1)+3(2\omega+1)(2\omega^2+1)+.....+(n+1)(n\omega+1)(n\omega^2+1)$ is **[Orissa JEE 2002]**

(a) $\left[\frac{n(n+1)}{2}\right]^2+n$ (b) $\left[\frac{n(n+1)}{2}\right]^2$ (c) $\left[\frac{n(n+1)}{2}\right]^2-n$ (d) None of these

Solution: (a) $2(\omega+1)(\omega^2+1)+3(2\omega+1)(2\omega^2+1)+.....+(n+1)(n\omega+1)(n\omega^2+1)=\sum_{r=1}^{n}(r+1)(r\omega+1)(r\omega^2+1)$

$=\sum_{r=1}^{n}(r+1)(r^2\omega^3+r\omega+r\omega^2+1)=\sum_{r=1}^{n}(r+1)(r^2-r+1)=\sum_{r=1}^{n}(r^3-r^2+r+r^2-r+1)=\sum_{r=1}^{n}(r^3)+\sum_{r=1}^{n}(1)=\left[\frac{n(n+1)}{2}\right]^2+n.$

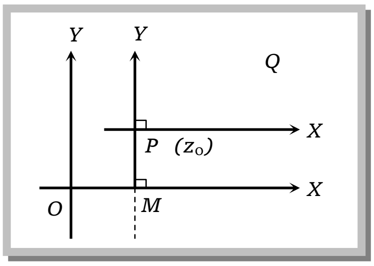
Example: 65 The roots of the equation $x^4-1=0$, are **[MP PET 1986]**

(a) $1,1,i,-i$ (b) $1,-1,i,-i$ (c) $1,-1,\omega,\omega^2$ (d) None of these

Solution: (b) Given equation $x^4-1=0 \Rightarrow (x^2-1)(x^2+1)=0 \Rightarrow x^2=1$ and $x^2=-1 \Rightarrow x=\pm1,\pm i$

2.18 Shifting the Origin in Case of Complex Numbers

Let O be the origin and P be a point with affix z_0. Let a point Q has affix z with respect to the co-ordinate system passing through O.



When origin is shifted to the point $P(z_0)$ then the new affix Z of the point Q with respect to new origin P is given by $Z=z-z_0$ *i.e.*, to shift the origin at z_0 we should replace z by $Z+z_0$.

Example: 66 If z_1, z_2, z_3 are the vertices of an equilateral triangle with z_0 as its circumcentre then changing origin to z_0, then (where z_1, z_2, z_3 are new complex numbers of the vertices)

(a) $z_1^2 + z_2^2 + z_3^2 = 0$ (b) $z_1 z_2 + z_2 z_3 + z_3 z_1 = 0$ (c) Both (a) and (b) (d) None of these

Solution: (a) In an equilateral triangle the circumcentre and the centroid are the same point. So,

$$z_0 = \frac{z_1 + z_2 + z_3}{3} \Rightarrow z_1 + z_2 + z_3 = 3z_0 \qquad \text{..... (i)}$$

To shift the origin at z_0, we have to replace z_1, z_2, z_3 and z_0 by $z_1 + z_0, z_2 + z_0, z_3 + z_0$ and $0 + z_0$ then equation (i) becomes $(z_1 + z_0) + (z_2 + z_0) + (z_3 + z_0) = 3(0 + z_0) \Rightarrow z_1 + z_2 + z_3 = 0$

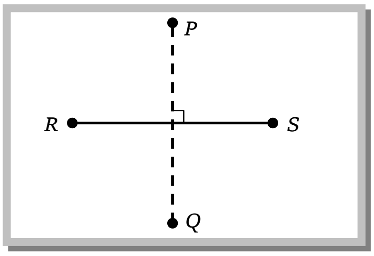
On squaring $z_1^2 + z_2^2 + z_3^2 + 2(z_1 z_2 + z_2 z_3 + z_3 z_1) = 0$ ..... (ii)

But triangle with vertices z_1, z_2 and z_3 is equilateral, then $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ .....(iii)

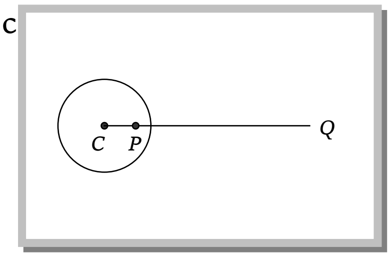
From (ii) and (iii) we get, $3(z_1^2 + z_2^2 + z_3^2) = 0$. Therefore, $z_1^2 + z_2^2 + z_3^2 = 0$.

2.19 Inverse Points

(1) **Inverse points with respect to a line :** Two points *P* and *Q* are said to be the inverse points with respect to the line *RS*. If *Q* is the image of *P* in *RS, i.e.,* if the line *RS* is the right bisector of *PQ*.



(2) **Inverse points with respect to a circle :** If *C* is the centre of the circle and *P,Q* are the inverse points with respect to the circle then three points *C,P,Q* are collinear, and also *CP* . *CQ* $= r^2$, where *r* is the radius of the c



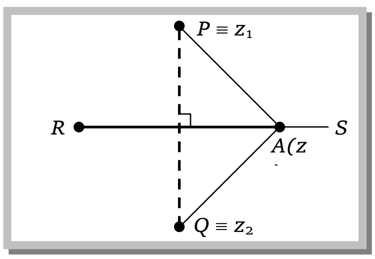
Example: 67 z_1, z_2, are the inverse points with respect to the line $z\bar{a} + a\bar{z} = b$ if

(a) $z_1 a + z_2 a = b$ (b) $z_1 \bar{a} + a\overline{z_2} = b$ (c) $z_1 \bar{a} - a\overline{z_2} = b$ (d) None of these

Solution: (b) Let *RS* be the line represented by the equation $z\bar{a} + a\bar{z} = b$

Let *P* and *Q* are the inverse points with respect to the line *RS*.

The point *Q* is the reflection (inverse) of the point *P* in the line *RS* if the line *RS* is the right bisector of *PQ*. Take any point *z* in the line *RS,* then lines joining *z* to *P* and *z* to *Q* are equal.



i.e., $|z-z_1|=|z-z_2|$ or $|z-z_1|^2=|z-z_2|^2$

i.e., $(z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2) \Rightarrow z(\bar{z}_2-\bar{z}_1)+\bar{z}(z_2-z_1)+(z_1\bar{z}_1-z_2\bar{z}_2)=0$ .....(ii)

Hence, equations (i) and (ii) are identical, therefore comparing coefficients, we get

$\frac{\bar{a}}{\bar{z}_2-\bar{z}_1}=\frac{a}{z_2-z_1}=\frac{-b}{z_1\bar{z}_1-z_2\bar{z}_2}$ So that, $\frac{z_1\bar{a}}{z_1(\bar{z}_2-\bar{z}_1)}=\frac{a\bar{z}_2}{\bar{z}_2(z_2-z_1)}=\frac{-b}{z_1\bar{z}_1-z_2\bar{z}_2}=\frac{z_1\bar{a}+a\bar{z}_2-b}{0}$

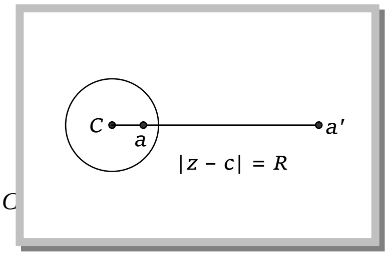
(By ratio and proportion rule)

Hence, $z_1\bar{a}+a\bar{z}_2-b=0$ or $z_1\bar{a}+a\bar{z}_2=b$.

Example: 68 Inverse of a point a with respect to the circle $|z-c|=R$ (a and c are complex numbers, centre C and radius R) is the point $c+\frac{R^2}{\bar{a}-\bar{c}}$

(a) $c+\frac{R^2}{\bar{a}-\bar{c}}$ (b) $c-\frac{R^2}{\bar{a}-\bar{c}}$ (c) $c+\frac{R}{\bar{c}-\bar{a}}$ (d) None of these

Solution: (a) Let a' be the inverse point of a with respect to the circle $|z-c|=R$, then by definition the points c, a, a' are collinear.



We have, $\arg(a'-c)=\arg(a-c)=-\arg(\bar{a}-\bar{c})$ $(\because \arg\bar{z}=-\arg z)$

$\Rightarrow \arg(a'-c)+\arg(\bar{a}-\bar{c})=0 \Rightarrow \arg\{(a'-c)(\bar{a}-\bar{c})\}=0$

$\therefore$ $(a'-c)(\bar{a}-\bar{c})$ is purely real and positive.

By definition $|a'-c||a-c|=R^2$ $(\because CP.C$

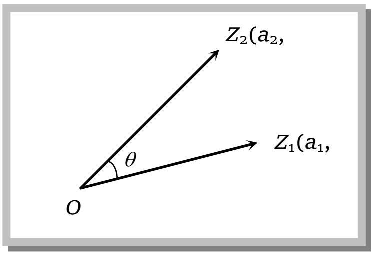
$\Rightarrow |a'-c||\bar{a}-\bar{c}|=R^2$ $(\because |z|=|\bar{z}|)$

$\Rightarrow |(a'-c)(\bar{a}-\bar{c})|=R^2 \Rightarrow (a'-c)(\bar{a}-\bar{c})=R^2$ $\{\because (a'-c)(\bar{a}-\bar{c})$ is purely real and positive$\}$

$\Rightarrow a'-c=\frac{R^2}{\bar{a}-\bar{c}}$. Therefore, the inverse point a' of a point a, $a'=c+\frac{R^2}{\bar{a}-\bar{c}}$.

2.20 Dot and Cross Product

Let $z_1=a_1+ib_1\equiv(a_1,b_1)$ and $z_2=a_2+ib_2\equiv(a_2,b_2)$ be two complex numbers.



If $\angle POQ=\theta$ then from coni method $\frac{z_2-0}{z_1-0}=\frac{|z_2|}{|z_1|}e^{i\theta}$

$\Rightarrow \frac{z_2\bar{z}_1}{z_1\bar{z}_1}=\frac{|z_2|}{|z_1|}e^{i\theta} \Rightarrow \frac{z_2\bar{z}_1}{|z_1|^2}=\frac{|z_2|}{|z_1|}e^{i\theta} \Rightarrow z_2\bar{z}_1=|z_1||z_2|e^{i\theta}$

$\Rightarrow z_2\bar{z}_1=|z_1||z_2|(\cos\theta+i\sin\theta)$

$\Rightarrow \text{Re}(z_2\bar{z}_1)=|z_1||z_2|\cos\theta$ .....(i) and $\text{Im}(z_2\bar{z}_1)=|z_1||z_2|\sin\theta$ .....(ii)

The dot product z_1 and z_2 is defined by $z_1 o z_2=|z_1||z_2|\cos\theta=\text{Re}(\bar{z}_1z_2)=a_1a_2+b_1b_2$ (From(i))

Cross product of z_1 and z_2 is defined by $z_1\times z_2=|z_1||z_2|\sin\theta=\text{Im}(\bar{z}_1z_2)=a_1b_2-a_2b_1$ (From(ii))

Hence, $z_1 o z_2=a_1a_2+b_1b_2=\text{Re}(\bar{z}_1z_2)$ and $z_1\times z_2=a_1b_2-a_2b_1=\text{Im}(\bar{z}_1z_2)$

Important Tips

☞ *If z_1 and z_2 are perpendicular then $z_1 o z_2=0$*

☞ *If z_1 and z_2 are parallel then $z_1\times z_2=0$*

☞ *Projection of* z_1 *on* $z_2 = (z_1 o z_2)/|z_2|$

☞ *Projection of* z_2 *on* $z_1 = (z_1 o z_2)/|z_1|$

☞ *Area of triangle if two sides represented by* z_1 *and* z_2 *is* $\frac{1}{2}|z_1 \times z_2|$

☞ *Area of a parallelogram having sides* z_1 *and* z_2 *is* $|z_1 \times z_2|$

☞ *Area of parallelogram if diagonals represents by* z_1 *and* z_2 *is* $\frac{1}{2}|z_1 \times z_2|$

Example: 69 If $z_1 = 2 + 5i, z_2 = 3 - i$ then projection of z_1 on z_2 is

(a) 1/10 (b) $1/\sqrt{10}$ (c) $-7/10$ (d) None of these

Solution: (b) Projection of z_1 on z_2 $= \frac{z_1 o z_2}{|z_2|}$ $= \frac{a_1 a_2 + b_1 b_2}{\sqrt{a_2^2 + b_2^2}} = \frac{1}{\sqrt{10}}$.

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